PART A TOPOLOGY
HT 2019
EXERCISE SHEET 2

For tutorials in week 5

Product topology, basis

Exercise 1. Consider $\mathbb{R}^n$ endowed with the norm $|| \ ||_\infty$.

(1) Prove that for any $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$, there is a $\mathbf{y} \in \mathbb{Q}^n$ and $\delta \in \mathbb{Q}$, $\delta > 0$, such that
$$\mathbf{x} \in B(\mathbf{y}, \delta) \subseteq B(\mathbf{x}, \varepsilon).$$

(2) Prove that the family $\{B(\mathbf{y}, \delta) : \mathbf{y} \in \mathbb{Q} \text{ and } \delta \in \mathbb{Q},\ \delta > 0\}$ is a countable basis for the Euclidean topology on $\mathbb{R}^n$.

Exercise 2. Prove that if $f\colon X \to Y$ is a continuous map of a space X to a Hausdorff space Y, then its graph
$$\mathcal{G}_f = \{(x, f(x)) \in X \times Y : x \in X\}$$
is a closed subset of $X \times Y$.

Connectedness

Exercise 3. (1) Let A and B be connected subsets of a topological space X such that $A \cap \overline{B} \neq \emptyset$. Prove that $A \cup B$ is connected.

(2) Which of the following subsets of $\mathbb{R}^2$ are connected?
- $B((1,0),1) \cup B((-1,0),1)$;
- $\overline{B((1,0),1)} \cup B((-1,0),1)$;
- the set of all points with at least one rational coordinate.

Exercise 4. Let X be a topological space.

(1) Let a be an arbitrary point in X. Prove that there exists a largest connected subset of X containing a, i.e. a set C_a such that:
- $a \in C_a$ and C_a is connected;
- for any connected subset S of X containing a, $S \subseteq C_a$.

We call such a set C_a *the connected component of X containing a*, or simply *a connected component of X*.

(2) Prove that C_a is closed for every $a \in X$.

(3) Prove that the relation $x \sim y \Leftrightarrow y \in C_x$ is an equivalence relation.

(4) Prove that connected components of X are either disjoint or they coincide.

(5) Find the connected components of $X = \{(x, y) \in \mathbb{R}^2 : x \neq y\}$ with the topology induced from $\mathbb{R}^2$.

Same question for $X = \{(z, w) \in \mathbb{C}^2 : z \neq w\}$ with the topology induced from $\mathbb{C}^2$.

(6) Let A be a subset of $\mathbb{R}$ such that $\mathring{A} = \emptyset$. Prove that the connected components of A are the singletons.

What are the connected components of $\mathbb{Q}$ with the topology induced from $\mathbb{R}$?

Exercise 5. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a differentiable function.

(1) Prove that the set $T = \{(x, y) \in I \times I : x < y\}$ is a connected subset of $\mathbb{R}^2$ with the standard topology.

(2) Let $g : T \to \mathbb{R}$ be the function defined by

$$g(x, y) = \frac{f(x) - f(y)}{x - y}.$$

Prove that $g(T) \subseteq f'(I) \subseteq \overline{g(T)}$.

(3) Show that $f'(I)$ is an interval.

Thus the derivative f' of any differentiable function $f : I \to \mathbb{R}$ always has the intermediate value property (without necessarily being continuous).

This is *Darboux's theorem.*

Compactness

Exercise 6. (1) Let X be a compact space, and let $(V_n)_{n\in\mathbb{N}}$ be a nested sequence of non-empty closed subsets of X (*nested* means that $V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}$). Prove that $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$.

(2) Now suppose that X is Hausdorff as well as compact, and let $f : X \to X$ be a continuous map. Let $X_0 = X$, $X_1 = f(X_0)$ and inductively define $X_{n+1} = f(X_n)$ for $n \geq 1$. Show that $A = \bigcap_n X_n$ is non-empty.

(3) Prove that $f(A) = A$.

[*Hint: the proof that $f(A) \subseteq A$ is straightforward. To show that any $a \in A$ is in $f(A)$, apply (1) to the sets $V_n = f^{-1}(a) \cap X_n$.*]

Exercise 7. Let X be a Hausdorff space and let A, B be disjoint compact subsets of X. Show that there exist disjoint open subsets U, V of X such that $A \subseteq U$ and $B \subseteq V$.