Chapter 6

Rotations, Angular Momentum, and Their Representations

In this chapter, we analyse a crucial symmetry that appears time and again in important quantum systems: that of three-dimensional rotations. You know well from your geometry course (and perhaps elsewhere) that the proper rotation group in three dimensions is $\mathrm{SO}(3)$, which can be identified with the group of three-by-three orthogonal matrices with unit determinant. In line with the general structures described in the previous chapter, we expect that for a quantum system describing objects in three-dimensions, there should be a (projective) unitary representation of $\mathrm{SO}(3)$ on our Hilbert space. We will have seen how this cashes out in practice by the end of the chapter.

6.1 Rotation group $\mathrm{SO}(3)$ and its infinitesimal generators

We begin with a review of some technical aspects of the the three-dimensional orthogonal group $\mathrm{O}(3)$. This group is normally realised as a group of three-by-three real matrices acting on Cartesian coordinates $x_i = (x_1, x_2, x_3)$ according to,

$$x_i \longrightarrow \sum_{j=1}^{3} R_{ij} x_j \;, \qquad RR^{\mathsf{T}} = \mathbb{1}_{3\times 3} \;, \tag{6.1}$$

where $\mathbb{1}_{3\times 3}$ is the three-by-three identity matrix. The special orthogonal group $\mathrm{SO}(3)$ restricts to those transformations that are rotations—it is the subgroup of $\mathrm{O}(3)$ for which $\det(R) = 1$.

As with translations, rotations can be taken arbitrarily close to the identity. To characterise this, let us consider a one-parameter family of rotation matrices $R(t)$ with $R(0) = \mathbb{1}_{3\times 3}$. (You may wish to think of this as the family of rotations about a fixed axis with t proportional to the angle of rotation.) We can define the matrix elements of an infinitesimal rotation matrix ω according to

$$R_{ij}(t) = \delta_{ij} + t\omega_{ij} + O(t^2) \;. \tag{6.2}$$

Expanding the condition $R(t)R^{\mathsf{T}}(t) = \mathbb{1}_{3\times 3}$ to first order in t gives

$$\omega_{ij} + \omega_{ji} = 0 \;, \tag{6.3}$$

or in matrix notation, $\omega + \omega^{\mathsf{T}} = 0$, *i.e.*, ω is a *skew symmetric* matrix. As you saw in prelims, it is natural to organise the components ω_{ij} of this matrix into a vector $\omega = (\omega_1, \omega_2, \omega_3) = (\omega_{32}, \omega_{13}, \omega_{21})$ that encodes the axis about which the instantaneous rotation is taking place and its magnitude. The vector and matrix index labelling for these parameters are related according to

$$\omega_i = -\frac{1}{2}\sum_{j,k} \varepsilon_{ijk}\omega_{jk} \;, \qquad \omega_{ij} = -\sum_k \varepsilon_{ijk}\omega_k \;. \tag{6.4}$$

The first-order action of $R(t)$ on the coordinate x_i then is given by

$$\begin{aligned} R(t)\mathbf{x} &= \mathbf{x} + t\delta\mathbf{x} + O(t^2) \;, \\ \delta x_i &= \sum_j \omega_{ij}x_j = \sum_j \varepsilon_{ijk}\omega_j x_k = (\omega \wedge \mathbf{x})_i \;. \end{aligned} \tag{6.5}$$

The group $\mathrm{SO}(3)$ is non-Abelian, so in general pairs of rotations do not commute, *i.e.*, $R\tilde{R} \neq \tilde{R}R$. This lack of commutativity is encoded in the *commutator* $R\tilde{R}R^{-1}\tilde{R}^{-1}$, which is itself element of $\mathrm{SO}(3)$ that will be the identity if and only if R and $\tilde{R}$ commute. Let us consider this commutator at the level of infinitesimal rotations. If we take t small in $R(t)$ and $\tilde{R}(t)$, then expanding the commutator to second order we have[38]

$$\begin{aligned} R(t)\tilde{R}(t)R(t)^{-1}\tilde{R}(t)^{-1} &= (1 + t\omega + \ldots)(1 + t\tilde{\omega} + \ldots)(1 - t\omega + \ldots)(1 - t\tilde{\omega} + \ldots) \;, \\ &= 1 + t^2(\omega\tilde{\omega} - \tilde{\omega}\omega) + \ldots \;, \end{aligned} \tag{6.6}$$

[38] You can feel free to take this equation for granted, but deriving it while keeping second-order terms might be instructive.

so here the noncommutativity manifests in terms of the matrix commutator $[\omega, \tilde{\omega}] = \omega\tilde{\omega} - \tilde{\omega}\omega$. Notice that

$$[\omega, \tilde{\omega}]_{ik} = \sum_j \omega_{ij}\tilde{\omega}_{jk} - \tilde{\omega}_{ij}\omega_{jk} = -\sum_l \varepsilon_{ikl}(\omega \wedge \tilde{\omega})_l \,, \tag{6.7}$$

where to prove this it is useful to use the identity $\sum_k \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. Alternatively, if we tacitly use (6.4) to identify vectors with skew-symmetric matrices,

$$[\omega, \tilde{\omega}] = (\omega \wedge \tilde{\omega}) \,. \tag{6.8}$$

The vector space of three-by-three skew-symmetric matrices endowed with the bilinear operation of the matrix commutator (observe that this preserves skew-symmetry) is known as the *Lie algebra* $\mathfrak{so}(3)$. In the theory of Lie groups, one finds that this matrix commutator encodes the full structure of the group $\mathrm{SO}(3)$ up to a single ambiguity, to which we will return later in our discussion of spin.

6.2 Rotations and wave functions

As our first example, we can define a an action of the rotation group on wave functions in three dimensions, *i.e.*, on the Hilbert space $L^2(\mathbb{R}^3)$, in a natural manner:

$$\begin{aligned} \mathrm{SO}(3) \times L^2(\mathbb{R}^3) &\longrightarrow L^2(\mathbb{R}^3) \,, \\ (R, \psi) &\longmapsto (U(R)\psi) \,, \qquad \begin{aligned} (U(R)\psi)\,(R\mathbf{x}) &= \psi(\mathbf{x}) \,, \\ (U(R)\psi)\,(\mathbf{x}) &= \psi(R^{\mathsf{T}}\mathbf{x}) \,. \end{aligned} \end{aligned} \tag{6.9}$$

The appearance of the transpose (*i.e.*, inverse) in the argument is analogous to the minus sign that we included in our translation operator, and analogously to that case we have for generalised position eigenstates,

$$U(R)\,|\mathbf{x}\rangle = |R\mathbf{x}\rangle \,. \tag{6.10}$$

This action is manifestly complex linear. It is also unitary, since we have

$$\langle U(R)\psi|U(R)\psi\rangle = \int_{\mathbb{R}^3} |\psi(R^{\mathsf{T}}\mathbf{x})|^2\,\mathrm{d}^3\mathbf{x} = \int_{\mathbb{R}^3} |\psi(\tilde{\mathbf{x}})|^2\,\mathrm{d}^3\tilde{\mathbf{x}} = \langle\psi|\psi\rangle \,, \tag{6.11}$$

where the change of variables $\mathbf{x} \to \tilde{\mathbf{x}} = R^{\mathsf{T}}\,\mathbf{x}$ introduces no Jacobian because R is an orthogonal matrix. Under composition, we see the importance of the transpose:[39]

$$(U(R_1)U(R_2)\psi)\,(\mathbf{x}) = (U(R_2)\psi)\,(R_1^{\mathsf{T}}\mathbf{x}) = \psi(R_2^{\mathsf{T}}R_1^{\mathsf{T}}\mathbf{x}) = \psi\left((R_1R_2)^{\mathsf{T}}\mathbf{x}\right) = (U(R_1R_2)\psi)\,(\mathbf{x}) \,. \tag{6.12}$$

so our operators satisfy the group law,

$$U(R_1R_2) = U(R_1)U(R_2) \,, \tag{6.13}$$

and we have a unitary representation of $\mathrm{SO}(3)$.

Let us consider the infinitesimal version of this action. Using the expansion for rotation matrices in Equation (6.2), we have

$$\begin{aligned} \psi(R^{\mathsf{T}}(t)\mathbf{x}) &= \psi\left(\mathbf{x} - t\omega \wedge \mathbf{x} + O(t^2)\right) \,, \\ &\approx \psi(\mathbf{x}) - t(\omega \wedge \mathbf{x}) \cdot \nabla\psi(\mathbf{x}) \,, \\ &= \psi(\mathbf{x}) - t\,\omega \cdot (\mathbf{x} \wedge \nabla\psi) \,, \\ &= \left(\mathbb{1}_{L^2(\mathbb{R})} - \frac{it}{\hbar}\omega \cdot \mathbf{L}\right)\psi(\mathbf{x}) \,. \end{aligned} \tag{6.14}$$

[39] You might try rewrite the manipulations in Equation (6.12) using bra-ket notations for wave functions to get a feeling for the way the compositions here are behaving and the relation to the action on generalised position eigenstates.

where $\mathbf{L}$ is the *orbital angular momentum* operator that you met in **A11 Quantum Theory**, which we can rewrite in terms of position and momentum operators,

$$\mathbf{L} := \mathbf{X} \wedge \mathbf{P} . \tag{6.15}$$

You have seen in that previous course, and one can compute explicitly, that the components L_i of the angular momentum operator obey the commutation relations

$$[L_i, L_j] = i\hbar \sum_k \varepsilon_{ijk} L_k . \tag{6.16}$$

For general vectors ω and $\tilde{\omega}$, one then finds

$$[\omega \cdot \mathbf{L}, \tilde{\omega} \cdot \mathbf{L}] = i\hbar(\omega \wedge \tilde{\omega}) \cdot \mathbf{L} , \tag{6.17}$$

or alternatively, in terms of the infinitesimal generators with extra constants included,

$$\left[-\frac{i}{\hbar}\omega \cdot \mathbf{L}, -\frac{i}{\hbar}\tilde{\omega} \cdot \mathbf{L}\right] = -\frac{i}{\hbar}(\omega \wedge \tilde{\omega}) \cdot \mathbf{L} , \tag{6.18}$$

We observe that these exactly match the commutation relation (6.8) with the replacement

$$\omega \quad \longleftrightarrow \quad -\frac{i}{\hbar}\omega \cdot \mathbf{L} , \tag{6.19}$$

where on the left hand side, ω represents a skew-symmetric matrix, and on the right hand side ω is a vector indicating the axis of rotation and we have operators on $L^2(\mathbb{R})$. We say that these operators furnish a *representation of the Lie algebra* $\mathfrak{so}(3)$ *on the Hilbert space* $L^2(\mathbb{R})$.

6.3 General unitary representations

In the previous analysis, we had a manifest action of the rotation group on the space of wave functions. In a more general and abstract setting, we must consider a general (projective) unitary representation of the rotation group on a Hilbert space $\mathcal{H}$. This isn't such an easy thing to get our hands on, so we will approach the problem through infinitesimal rotations. We introduce infinitesimal generators of rotations (in the sense of Stone's theorem) and denote them by $\mathbf{J}$. For a one-parameter families of rotations $R(t)$, we then have (just as we did for wave functions),[40]

$$U(R(t)) = 1_{\mathcal{H}} - \frac{it\, \omega \cdot \mathbf{J}}{\hbar} + O(t^2) . \tag{6.20}$$

We can compare the group-theoretic commutator of two rotations with the composition taken both before and after applying the map to $\mathsf{U}(\mathcal{H})$; we have the equation

$$U(R(t))\, U\left(\tilde{R}(t)\right)\, U(R(t))^*\, U\left(\tilde{R}(t)\right)^* = U\left(R(t)\tilde{R}(t)R(t)^{-1}\tilde{R}(t)^{-1}\right) , \tag{6.21}$$

where on the left we have the commutator of elements of $\mathsf{U}(\mathcal{H})$, and on the right we have the image in $\mathsf{U}(\mathcal{H})$ of the commutator of elements of $\mathsf{SO}(3)$. Letting each rotation be infinitesimal of the same order, we get, by comparing terms at second order,

$$[\omega \cdot \mathbf{J}, \tilde{\omega} \cdot \mathbf{J}] = i\hbar(\omega_{(1)} \wedge \omega_{(2)}) \cdot \mathbf{J} \tag{6.22}$$

which is exactly analogous to (6.18) with $\mathbf{L}$ replaced by $\mathbf{J}$. In components, this is

$$[J_i, J_j] = i\hbar \sum_k \varepsilon_{ijk} J_k . \tag{6.23}$$

This is an important result; whenever we have a representation of the rotation group on a Hilbert space, we get a trio of self-adjoint angular momentum operators, $\{J_i\}$, that obey the commutation relations (6.23) and generate the action

[40] For the very discerning reader, the linear dependence on ω in (6.20) requires some explanation.

of more general rotations via exponentiation in the sense of Stone's theorem. This is an instance of a fundamental relationship between representations of Lie groups and representations of Lie algebras.

6.4 Angular momentum multiplets

We have (at least partially) reduced problem of studying of rotations in quantum systems to the study of *representations of the angular momentum operators*:

Definition 6.4.1. A *representation of the angular momentum operators* is a Hilbert space, $\mathcal{H}$, equipped with an action of three self-adjoint operators $J_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, 3$, satisfying the commutation relations (6.23).

Remark 6.4.2. This is equivalent to a representation of the Lie algebra $\mathfrak{so}(3)$ on $\mathcal{H}$. The difference is in the factor of $\hbar$ on the right hand side of (6.23), which can be removed by an appropriate rescaling of the J_i. Also, in some cases it is conventional for a representation of $\mathfrak{so}(3)$ to be described in terms of anti-self adjoint operators (operators obeying $A^* = -A$), in which case a factor of i is incorporated into the rescaling as well.

Definition 6.4.3. An *irreducible representation* of the angular momentum operators is a representation of the angular momentum operators for which there is no a proper subspace $\mathcal{H}_{\text{sub}} \subset \mathcal{H}$ with $J_i : \mathcal{H}_{\text{sub}} \to \mathcal{H}_{\text{sub}}$, *i.e.*, $\mathcal{H}$ contains no proper sub-representation of the angular momentum operators.

In **A11 Quantum Theory**, in the context of discussing orbital angular momentum for three-dimensional wave functions, you identified the structure of general irreducible representations of the angular momentum operators. Here we will recall the story in the general case. We define the *total angular momentum operator* $\mathbf{J}^2 = \mathbf{J} \cdot \mathbf{J}$. A short calculation shows that

$$[\mathbf{J}^2, J_i] = 0 \;, \tag{6.24}$$

so the action of the J_i operators preserves eigenspaces of $\mathbf{J}^2$. Since $\mathbf{J}^2$ is self-adjoint, we can choose to work in a basis of its eigenstates for any representation of the angular momentum operators, and so if $\mathcal{H}$ is an irreducible representation, then $\mathbf{J}^2$ must just act by a multiple of the identity on $\mathcal{H}$. We can give a completely explicit description of all finite-dimensional, irreducible representations if we furthermore choose to diagonalise J_3.

Theorem 6.4.4. The irreducible representations of the angular momentum operators are labeled by a non-negative half-integer $j = 0, \frac{1}{2}, 1, \ldots \in \frac{1}{2}\mathbb{N}$ known as the *spin* of the representation. Denote the Hilbert space admitting such a representation by $\mathcal{H}_{spin\, j}$. The dimension of $\mathcal{H}_{spin\, j}$ is $2j+1$ and $\mathbf{J}^2$ acts with eigenvalue $\hbar^2 j(j+1)$.

There is an orthonormal basis of $\mathcal{H}_{spin\, j}$ consisting of eigenvectors $|j,m\rangle$ of J_3 with $J_3\,|j,m\rangle = \hbar m\,|j,m\rangle$ for $m = -j, -j+1, \ldots j-1, j$.

Proof. We introduce the ladder operators $J_\pm = J_1 \pm iJ_2$, which commute with $\mathbf{J}^2$. We also can check that

$$[J_3, J_\pm] = \pm\hbar J_\pm \;. \tag{6.25}$$

This gives them the interpretation as raising and lowering operators for eigenvectors $|j,m\rangle$ of J_3 (with eigenvalue $\hbar m$, say):

$$J_3\left(J_\pm\,|j,m\rangle\right) = \pm\hbar J_\pm\,|j,m\rangle + J_\pm J_3\,|j,m\rangle = \hbar(m\pm 1)\left(J_\pm\,|j,m\rangle\right) \;. \tag{6.26}$$

Thus $J_\pm\,|j,m\rangle$ is a multiple of an eigenvector for J_3 with eigenvalue $\hbar(m\pm1)$. The following then shows that the values for $|m|$ must be bounded.

Lemma 6.4.5. Let $\mathbf{J}^2\,|\psi\rangle = \lambda\hbar^2\,|\psi\rangle$ and $J_3\,|\psi\rangle = \hbar m\,|\psi\rangle$. Then for all $\phi \in \mathcal{H}$,

$$\langle J_\pm\phi|J_\pm\psi\rangle = \hbar^2\left(\lambda - m(m\pm1)\right)\langle\phi|\psi\rangle \quad \text{and} \quad \|J_\pm\psi\|^2 = \hbar^2\left(\lambda - m(m\pm1)\right)\|\psi\|^2 \;. \tag{6.27}$$

Proof. Observe from the angular momentum commutation relations that

$$J_+J_- = \mathbf{J}^2 - J_3^2 + \hbar J_3 \;, \qquad J_-J_+ = \mathbf{J}^2 - J_3^2 - \hbar J_3 \;, \tag{6.28}$$

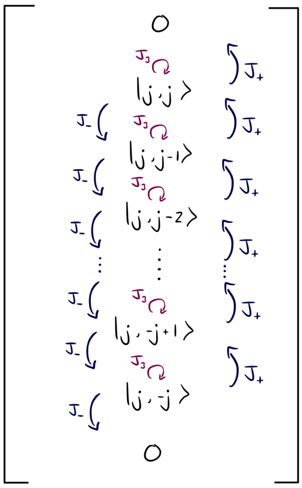

Figure 2. Depiction of irreducible representation of the angular momentum operators.

so the identities follow from

$$\langle J_-\phi|J_-\psi\rangle = \langle\phi|J_+J_-\psi\rangle = \big\langle\phi\big|(\mathbf{J}^2 - J_3^2 + \hbar J_3)\psi\big\rangle \ , \tag{6.29}$$

and using the eigenvalue relations (and similarly for the J_+ plus version of (6.29)). ■

Given that λ is fixed on an irreducible representation, $|m|$ cannot be too large as otherwise the norm squared of these states would be negative. The only way to avoid $|m|$ becoming arbitrarily large in the negative direction is if for some smallest value m_-, $J_-|\psi_{m_-}\rangle = 0$ where $J_3|\psi_{m_-}\rangle = \hbar m_-|\psi_{m_-}\rangle$, which requires $\lambda = m_-(m_- - 1)$. The only way that $|m|$ can avoid becoming arbitrarily large in the positive direction is if analogously for some largest value m_+, $J_+|\psi_{m_+}\rangle = 0$, so $\lambda = m_+(m_+ + 1)$. To realise both situations at once, we need

$$\lambda = j(j+1)\ , \qquad m_- = -j\ , \qquad m_+ = j\ . \tag{6.30}$$

By construction $m_+ - m_- = 2j$ must be an integer (since starting with the $|\psi_{m_-}\rangle$ and acting repeatedly with J_+ we must arrive eventually at $|\psi_{m_+}\rangle$). Hence the constraints on the eigenvalues are as stated in the theorem.

To finish off the proof, we require that the J_3 eigenvalues be nondegenerate. This follows from irreducibility. Suppose that there are two linearly independent eigenvectors $|j,m;1\rangle$ and $|j,m;2\rangle$ that, without loss of generality, can be taken to be mutually orthogonal. Then it follows from the expressions above that $J_\pm^n\,|j,m;1\rangle$ and $J_\pm^n\,|j,m;2\rangle$ are orthogonal. Thus there will be two nontrivial J_i-invariant subspaces spanned by $J_\pm^n\,|j,m;1\rangle$ and by $J_\pm^n\,|j,m;2\rangle$, contradicting irreducibility. ■

We conclude with a few additional comments:

- If we are working in a definite irreducible representation of spin j, we might sometimes simply denote the state kets $|m\rangle$ to encode the J_3 eigenvalue.
- The basis $|j,m\rangle$ of $\mathcal{H}_{spin\,j}$ is unique up to an *overall normalisation* for the entire representation if we impose the normalisation conditions
$$|j,m\pm 1\rangle = \frac{J_\pm\,|j,m\rangle}{\hbar\sqrt{j(j+1) - m(m\pm 1)}}\ . \tag{6.31}$$
This definition ensures in particular that the states $|j,m\rangle$ all have the same norm, so if we choose a particular state, say $|j,j\rangle$, to be unit normalised and construct the rest of the representation by the action of J_-, then all of these states will be unit normalised.

- Important examples of representations with integer spin were given in **A11 Quantum Theory** in terms of spherical harmonics. These are angular momentum representations realised using the orbital angular momentum operators **L**, which can be written in spherical polar coordinates as

$$L_\pm = i\hbar\, e^{\pm i\varphi}\left(\cot\theta\frac{\partial}{\partial\varphi} \pm i\frac{\partial}{\partial\theta}\right) , \qquad L_3 = -i\hbar\frac{\partial}{\partial\varphi} . \tag{6.32}$$

The total spin j is usually denoted by ℓ in this context and is required to be an integer. The wave functions $\Psi_\ell^m(\varphi,\theta)$ corresponding to the basis states $|\ell, m\rangle$ take the form

$$Y_\ell^m(\varphi,\theta) = P_\ell^m(\cos\theta)e^{im\varphi} , \tag{6.33}$$

where $P_\ell^m(x)$ are associated Legendre functions. The requirement that ℓ and m be integral follows from the need for $e^{im\varphi}$ to be single valued.

6.5 Spin $1/2$

We saw that while half-integral spin is acceptable in the context of representations of the angular momentum operators, it doesn't arise in the context of orbital angular for three-dimensional wave functions. Let us investigate the simplest case: spin $j = 1/2$.

The discussion above gives an explicit realisation of this representation,

$$\mathcal{H}_{\text{spin }\frac{1}{2}} \cong \mathbb{C}^2 = \text{Span}\left\{ \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle , \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \right\} . \tag{6.34}$$

Of course this is just our old friend the qubit. The above action of $J_\pm$ and hence J_1 and J_2 is determined by (6.31) for which in this case the denominator is just $\hbar$, and the eigenvalue condition determines J_3. It follows that in this basis we have

$$\mathbf{J} = \frac{\hbar}{2}\sigma , \tag{6.35}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the same Pauli spin matrices we met in our qubit discussion. Now let us consider a general rotation by some angle θ about an axis designated by the unit vector $\mathbf{n}$; we denote this by $R_\mathbf{n}(\theta)$. By Stone's theorem, this should be realised on our two-dimensional Hilbert space by the unitary matrix

$$U(R_\mathbf{n}(\theta)) =: U_\mathbf{n}(\theta) = \exp\left(-\frac{i\theta}{\hbar}\mathbf{n}\cdot\mathbf{J}\right) = \exp\left(-\frac{i\theta}{2}\mathbf{n}\cdot\sigma\right) . \tag{6.36}$$

An explicit computation of this matrix exponential yields a simple expression for the matrix that should represent the rotation,

$$U_\mathbf{n}(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{1}_{2\times2} - i\sin\left(\frac{\theta}{2}\right)\mathbf{n}\cdot\sigma . \tag{6.37}$$

It is easy to confirm that these are unitary matrices, and in addition they are manifestly traceless, so are elements of SU(2). Indeed, by letting $\mathbf{n}$ range over the unit sphere in three dimensions and letting θ run from 0 to 2π, this gives a parameterisation of the most general element of SU(2). However, compared to rotations this is double counting! Rotating by θ around the axis defined by $\mathbf{n}$ is the same as rotating by $2\pi - \theta$ around the axis defined by $-\mathbf{n}$.

Indeed, for fixed $\mathbf{n}$, we see that setting $\theta = 2\pi$ doesn't give us back the identity, but rather minus the identity. It is only upon taking $\theta = 4\pi$ that our unitary matrix returns to the identity. So there is a two-to-one correspondence between the elements of SU(2) and the inequivalent rotations, *i.e.*, the elements of SO(3).

We are encountering in this example precisely a situation where our symmetry group (SO(3)) is implemented via a projective unitary representation that is not strictly a unitary representation of the group we started with. We can see this in terms of the group law. Consider the rotation $R_\mathbf{n}(\pi)$ that performs a half rotation about the axis $\mathbf{n}$. Then performing this twice we have have

$$U\left(R_\mathbf{n}(\pi)\right) U\left(R_\mathbf{n}(\pi)\right) = U_\mathbf{n}(2\pi) = -\mathbb{1}_{2\times2} . \tag{6.38}$$

whereas if we compose the rotations before taking the map to unitary matrices, we have

$$U(R_{\mathbf{n}}(\pi)R_{\mathbf{n}}(\pi)) = U(R_{\mathbf{n}}(2\pi)) = U(R_{\mathbf{n}}(0)) = 1_{2\times 2} . \tag{6.39}$$

The sign difference is precisely the type of "extra phase" that is allowed for projective representations!

It turns out that this example is indicative of the general story for half-integer-spin representations. These are projective unitary representations of $\mathrm{SO}(3)$ that do not lift to unitary representations of $\mathrm{SO}(3)$. Rather, they correspond to unitary representations of $\mathrm{SU}(2)$, where the relation between the two groups is by a quotient,

$$\mathbb{P}\,\mathrm{SU}(2) := \mathrm{SU}(2)/\{\pm 1\} \cong \mathrm{SO}(3) . \tag{6.40}$$

In the case of orbital angular momentum, there is *manifestly* a representation of the honest rotation group via the action on wave functions; consequently only integer spin can occur.

Remark 6.5.1. There is a beautiful observation to make here that I cannot help but include for your entertainment (I hope). As was observed above, we have a realisation of $\mathrm{SU}(2)$ by a choice of unit vector in $\mathbb{R}^3$ and an angle $\theta \in 2\pi$. This gives us a realisation of $\mathrm{SU}(2)$ as a *circle fibration over the two-sphere* (you can imagine a circle corresponding to the choice of angle sitting over each point on the two-sphere corresponding to the choice of unit vector). This is what's known as the *Hopf fibration*, which realises the three sphere $S^3 \cong \mathrm{SU}(2)$ as a circle fibration over S^2. The rotation group $SO(3)$ then gets identified as the quotient space $S^3/\mathbb{Z}_2$, with $\mathbb{Z}_2$ acting as the antipodal map.