STRING THEORY I

Lecture 5

Chapter 3 Old covariant quantization

Classical theory: field theory on the 2dim world-sheet

$$S_p = -\frac{T}{2}\int_\Sigma d\tau\, d\sigma \left(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right)$$

in the conformal unit gauge $\gamma_{ab}=\eta_{ab}$.

This is supplemented by the constraints

$$T_{++}=0 \quad \& \quad T_{--}=0 .$$

The OCQ approach consists on promoting the canonical fields X^μ & their conjugate momenta $\Pi^\mu_\tau = T\partial_\tau X^\mu$ to operators and the Poisson brackets $\{\cdot,\cdot\}_{PB}$ to commutators of operators

$$\{\cdot,\cdot\}_{PB} \rightsquigarrow i[\cdot,\cdot]$$

We get the canonical equal τ time commutation relations

$$[\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = -i\,\delta(\sigma-\sigma')\,\eta^{\mu\nu}$$

(with $[X^\mu(\sigma), X^\nu(\sigma')] = 0$, $[P^\mu(\sigma), P^\nu(\sigma')] = 0$)

The operators X^μ & Π^μ are hermitian

$$X^\mu = (X^\mu)^\dagger, \quad \Pi^\mu = (\Pi^\mu)^\dagger$$

The commutation relations for the oscillator modes follow immediately from this:

$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu} \qquad \text{(Heisenberg algebra)}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

The hermiticity of the operators X^μ & Π^μ imply that the two mode operators $\hat{p}^\mu, \hat{x}^\nu$ are Hermitian while

$$\alpha^\mu_{-m} = (\alpha^\mu_m)^\dagger$$

$$\tilde{\alpha}^\mu_{-m} = (\tilde{\alpha}^\mu_m)^\dagger$$

The set $\{\alpha_m^\mu, \tilde{\alpha}_m^\mu\}$ is an infinite set of harmonic oscillators raising & lowering operators.

We think of

$$\alpha_m^\mu \quad (\tilde{\alpha}_m^\mu) \quad m>0 \text{ as } \quad \text{annihilation ops}$$

$$\alpha_{-m}^\mu \quad (\tilde{\alpha}_{-m}^\mu) \quad m>0 \text{ as } \quad \text{creation ops}$$

Now we can construct the Hilbert space

Hilbert space:

1. Define the oscillator vacuum state $|0\rangle_{vac}$

$$\alpha_m^\mu |0\rangle_{vac} = \tilde{\alpha}_m^\mu |0\rangle_{vac} = 0 \quad \forall m \geq 1$$

2. Next, we build the oscillator Fock spaces:

$$\mathcal{H}_{open}^{Fock} = \mathrm{span}\Big\{ \overbrace{\prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i} |0\rangle_{vac}}^{|0\rangle,\ \alpha_{-n}^{\mu}|0\rangle,\ \dots} \Big\}_{n_i \geq 1}$$

$$\mathcal{H}_{closed}^{Fock} = \mathrm{span}\Big\{ \prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i} \prod_{j=1}^{l} \tilde{\alpha}_{-n_j}^{\mu_j} |0\rangle_{vac} \Big\}_{n_i, n_j \geq 1} = \underbrace{\mathcal{H}_R^{Fock}}_{\cong\, \mathcal{H}_{open}} \otimes \underbrace{\mathcal{H}_L^{Fock}}_{\cong\, \mathcal{H}_{open}}$$

Introduce oscillator number operator

$$N = \sum_{k>0} \alpha_{-n} \cdot \alpha_n \qquad \tilde{N} = \sum_{k>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n$$

which count oscillators

$$N \left(\prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i} |0\rangle_{vac} \right) = \left(\sum_{i=1}^{k} n_i \right) \left(\prod_{i=1}^{k} \alpha_{-n_i}^{k} |0\rangle_{vac} \right)$$

$$N \left(\prod_{i=1}^{k} \tilde{\alpha}_{-n_i}^{\mu_i} |0\rangle_{vac} \right) = \left(\sum_{i=1}^{k} n_i \right) \left(\prod_{i=1}^{k} \tilde{\alpha}_{-n_i}^{k} |0\rangle_{vac} \right)$$

$$N \alpha_{-n}^{\mu} |0\rangle = \sum_{m\geq 1} \left(\alpha_{-n} \alpha_m \right) \alpha_{-n}^{\mu} |0\rangle =$$

$$= \sum_{m\geq 1} \alpha_{-m} \cdot \left(\underbrace{[\alpha_m, \alpha_{-n}^{\mu}]}_{n\,\delta_{m-n,0}} + \cancel{\alpha_{-n}^{\mu} \alpha_m} \right) |0\rangle = n\, \alpha_{-n}^{\mu} |0\rangle$$

One can organize the oscillator states into levels, ie states with a given N ($\hat{N}$) eigenvalue.

For open strings (or R-movers, closed string)

- $N=0$ $\quad |0\rangle_{vac}$
- $N=1$ $\quad \alpha_{-1}^{\mu}|0\rangle_{vac}$
- $N=2$ $\quad \alpha_{-2}^{\mu}|0\rangle_{vac}$, $\alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}|0\rangle_{vac}$
- $N=3$ $\quad \alpha_{-3}^{\mu}|0\rangle_{vac}$, $\alpha_{-2}^{\mu_1}\alpha_{-1}^{\mu_2}|0\rangle_{vac}$, $\alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}\alpha_{-1}^{\mu_3}|0\rangle_{vac}$

$\vdots$

Hence $$\mathcal{H}^{Fock} = \text{Span}\left\{ \prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i} |0\rangle_{vac} \right\}_{n_i \geq 1} = \bigoplus_{N=0}^{\infty} \mathcal{H}^{Fock}[N]$$

(similar for oscillators $\tilde{\alpha}$)

3 zero-modes

But we have not yet completely specified the states as we still have the zero-modes.

The ground state is in fact $\Psi(x)\,|0\rangle_{vac}$

is a spatial wave function

In momentum space we can choose states to be eigenstates of the center of mass momentum operator $\hat{p}^\mu$

$$\hat{p}^\mu |K\rangle = K^\mu |K\rangle , \quad K^\mu \in \mathbb{R}^{1,D-1}$$

obeying

$$\langle K' | K \rangle = \delta^{(D)}(K-K')$$

$$\therefore \quad \mathcal{H}_{\text{zero-mode}} \simeq L^2(\mathbb{R}^{1,D-1})$$

The Hilbert space is then

$$\mathcal{H}_{\text{open}} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{\text{Fock}}_{\text{open}}$$

$$\mathcal{H}_{\text{closed}} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{\text{Fock}}_{L} \otimes \mathcal{H}^{\text{Fock}}_{R}$$

state $\prod_{i=1}^{n} \alpha^{\mu}_{-n_i} |0, p\rangle$ $\qquad N = \sum_{i=1}^{n} n_i$

momentum eigenvalue p^{μ}

closed string $\prod \alpha \cdot \prod \tilde{\alpha} \, |0, p\rangle$

Problem (or not?) : Consider the state

$$|\varphi\rangle = \alpha^0_{-1}|0;k\rangle$$

Then

$$\langle\varphi|\varphi\rangle = \langle 0;k|\alpha^0_{+1}\alpha^0_{-1}|0;k'\rangle = \eta^{00}\,\delta(k-k')$$

$$[\alpha^0_{+1},\alpha^0_{-1}]$$

$$= -\delta(k-k')$$

Wrong sign! There are negative norm states

ghosts!

However: we have not imposed the constraints

Normal ordering:

Recall the Witt-generators

$$L_m = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{m-k} \cdot \alpha_k \,, \qquad L_m^\dagger = L_{-m} \quad (m \neq 0)$$

$$\tilde{L}_m = \frac{1}{2} \sum_{k=-\infty}^{\infty} \tilde{\alpha}_{m-k} \cdot \tilde{\alpha}_k \,, \qquad \tilde{L}_m^\dagger = \tilde{L}_{-m} \quad (m \neq 0)$$

are quadratic in the oscillators.

The operators α_{m-k} & α_k commute $\forall k$ unless $m=0$ so the only "problematic" operator is L_0 (similar for $\tilde{L}_0$).

Suppose we choose naively

$$L_0 = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{-k} \cdot \alpha_k = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \frac{1}{2} \sum_{k \geq 1} \alpha_{-k} \cdot \alpha_k + \frac{1}{2} \sum_{k \geq 1} \alpha_k \cdot \alpha_{-k}$$

Then (open string)

$\alpha_0^\mu = p^\mu \ell$

$$L_0 |0;p\rangle = \left(\frac{\ell^2 p^2}{2} + \frac{1}{2} \sum_{k \geq 1} \alpha_k \cdot (\alpha_{-k} |0,p\rangle) \right)$$

$[\alpha_k^\mu, \alpha_{-k}^\nu] \eta_{\mu\nu}$

$[\alpha_k^\mu, \alpha_{-k}^\nu] = k \eta^{\mu\nu}$

$$= \left(\frac{\ell^2 p^2}{2} + \frac{1}{2} \sum_{k \geq 1} k D \right) |0,p\rangle$$

$$= \left(\frac{\ell^2 p^2}{2} + \frac{1}{2} D \sum_{k \geq 1} k \right) |0,p\rangle$$

Zeta function regularization

$$\sum_{k \geq 1} k = \zeta(-1) = -\frac{1}{12}$$

$$= \left(\frac{\ell^2 p^2}{2} - \frac{1}{24} D \right) |0,p\rangle$$

so maybe not!

To account properly for the normal ordering one can try to define the quantum operator as

$$L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \underbrace{:\alpha_{-n} \cdot \alpha_n:}_{}$$

normal ordered product
(lowering ops to the right)

by

closed strings
$$\begin{cases} L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n \geq 1} \alpha_{-n} \cdot \alpha_n = \frac{\ell^2 p^2}{8} + N \\ \tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0 \cdot \tilde{\alpha}_0 + \sum_{n \geq 1} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = \frac{\ell^2 p^2}{8} + \tilde{N} \end{cases}$$

$$\Rightarrow L_0 |0;p\rangle = \frac{\ell^2 p^2}{8} |0;p\rangle$$

open strings
$$L_0 = \frac{\ell^2 p^2}{2} + N$$

Virasoro algebra We need to check the commutator algebra of the operators L_m ($\tilde{L}_m$)

PS 2: a direct computation gives

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12}(m^3-m)\,\delta_{m+n,0}}$$

(similar for $[\tilde{L}_m, \tilde{L}_n]$ for closed strings)

This is the Virasoro algebra with central charge D.

The Virasoro algebra is a central extension of the Witt-algebra

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathrm{Vir} \arrow[r] & \mathrm{Witt} \arrow[r] & 0 \\ & \hat{c}\,;\ [\hat{c}, L_n] = 0 \arrow[u] & \hat{c} = 0 \arrow[u, squiggly] & \mathrm{Vir}/\langle \hat{c} \rangle \arrow[u, squiggly] & \end{tikzcd}$$

The central extension is related to an anomaly of the Weyl invariance (more later).

The global $sl(2)$ algebra generated by $\{L_0, L_1, L_{-1}\}$ (or $\{\tilde{L}_0, \tilde{L}_1, \tilde{L}_{-1}\}$) is not anomalous.

Remark: An alternative way to deal with the normal ordering is to replace the naive operator L_0 above with

$$L_0 - a$$

where a = constant. This also gives the central charge term, see GSW.

Constraints

Impose the constraints in the quantum theory to identify the physical states

Another problem arises due to appearance of the central charge $\boxed{\text{if}}$ we define the physical states as the states $|\psi\rangle$ which satisfy

$$L_m |\psi\rangle = 0 \qquad \forall m$$

In fact, there is a contradiction.

Consider $\quad [L_m, L_n]|\psi\rangle$

$\uparrow$ physical state $\quad L_m|\psi\rangle = 0 \quad \forall m$

(a) $\quad [L_m, L_n]|\psi\rangle = (L_m L_n - L_n L_m)|\psi\rangle = 0 \quad \forall m, n$

(b) $\quad [L_m, L_n]|\psi\rangle = \Big[\underbrace{(m-n)\cancel{L_{m+n}} + \frac{D}{12}(m^3-m)\delta_{m+n,0}}_{\neq 0 \;\; m=-n}\Big]|\psi\rangle$

But for $\quad n=-m$

$$[L_m, L_{-m}]|\psi\rangle = \frac{D}{12}(m^3-m)\,|\psi\rangle$$

Imposing all constraints leads to a $\boxed{\text{trivial}}$ Hilbert space when $\boxed{D \neq 0}$!

Instead we define physical states $\phi, \psi \in \mathcal{H}_{phys}$ by the constraints $\langle \psi | L_m | \phi \rangle = 0 \quad \forall m \neq 0$

Definition: a state $|\phi\rangle$ is physical if

$\{L_m, m \geq 1\}$ subalgebra $\rightarrow$

- $L_m |\phi\rangle = 0 \qquad \forall\, m \geq 1$
- $(L_0 - a)|\phi\rangle = 0 \qquad$ for fixed a

normal ordering constant.

Virasoro constraints or physical state conditions

Remark: $\langle \psi | L_m | \phi \rangle = 0 \qquad \forall\, m \geq 1$

$$\langle \psi | L_m | \phi \rangle = \langle \phi | L_{-m} | \psi \rangle^* = 0 \qquad \forall\, m \leq -1$$

$$\langle \phi | (L_0 - a) | \psi \rangle = 0$$

Important remark: the generators of D-dim spacetime Poincaré symmetries have no normal ordering ambiguities as can be seen from

$$P^\mu = p^\mu$$

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right)$$

open strings

$$M^{\mu\nu} = \int d\sigma (X^\mu P^\nu - P^\nu X^\mu)$$

Moreover:
$$[P^\mu, L_m] = [M^{\mu\nu}, L_m] = 0 \qquad \forall m$$

$$[L_m, \alpha_n^\mu] = -n\,\alpha_{m+n}^\mu$$

P^μ & $M^{\mu\nu}$ preserve the physical state conditions. This means that the phys. decomposes into representations of $ISO(1, D-1)$ (covariant quantization).

Massshell & level-matching conditions: the L_0 (& $\tilde{L}_0$) condition

Closed string

mass shell condition

$$(L_0 + \tilde{L}_0 - 2a)|\varphi, K\rangle = 0 \Leftrightarrow \left(\frac{\ell^2 k^2}{4} + N + \tilde{N} - 2a\right)|\varphi, K\rangle = 0$$

so $$\frac{\ell^2 M^2}{4} = N + \tilde{N} - 2a \Rightarrow \alpha' M^2 = 2(N + \tilde{N} - 2a)$$

$\alpha' = \ell/2$

level matching condition

$$(L_0 - \tilde{L}_0)|\varphi, K\rangle = 0 \Leftrightarrow (N - \tilde{N})|\varphi, K\rangle = 0 \quad \text{so} \quad N = \tilde{N}$$

Open string

mass shell condition

$$(L_0 - a)|\varphi, K\rangle = 0 \Leftrightarrow \left(\frac{\ell^2 p^2}{2} + N - a\right)|\varphi, K\rangle = 0 \quad \text{so} \quad \alpha' M^2 = N - a$$

Level 0 ground $|0, k\rangle$

As an exercise, you can show that

$$[N, L_m] = -m L_m$$

so L_m shifts N-level by $-m$ (similar for $\tilde{N}$ & $\tilde{L}$)

$\Rightarrow$ at level zero we only need to check the L_0 conditions.

The L_0-conditions at level $N=0$ are

open strings $\qquad \alpha' M^2 = -a$

closed strings $\qquad \alpha' M^2 = -4a$

$\therefore$

$a < 0$ massive ground state

$a = 0$ massless ground state

$a > 0$ tachyonic ground state (!)

Level one states & dealing with ghosts

Recall: earlier we encountered a problem with negative norm states (ghosts) at level one of the open string (eg: $\alpha^0_{-1}|0;K\rangle$). We want to see if this issue remains after applying the constraints

Consider a general level one open string state

$$\zeta\cdot\alpha_{-1}|0;K\rangle = |\zeta;K\rangle \qquad \zeta\in\mathbb{R}^{1,D-1}$$

polarization vector

and impose the physical state conditions for L_0 & L_{+1}.

The L_n conditions for $n\geq 2$ are satisfied automatically once the L_0 & L_{+1} are imposed.

- mass-shell: $-\alpha' k^2 = \alpha' M^2 = 1 - a$
- L_{+1} condition: $L_{+1}(\zeta\cdot\alpha_{-1})|0;k\rangle = \eta_{\mu\nu}\zeta^\mu[L_{+1},\alpha_{-1}^\nu]|0;k\rangle$

$$= \eta_{\mu\nu}\zeta^\mu \alpha_0^\nu |0;k\rangle = \ell(\zeta\cdot k)|0;k\rangle$$

$$L_{+1}|\zeta;k\rangle = 0 \iff \zeta\cdot k = 0$$

OTOH

The norm of a general level 1 state $|\zeta,k\rangle$ is

$$\langle\zeta;k|\zeta';k'\rangle = \langle 0;k|(\zeta\cdot\alpha_{+1})(\zeta'\cdot\alpha_{-1})|0;k'\rangle$$

$$= (\zeta\cdot\zeta')\langle 0;k|0;k'\rangle = (\zeta\cdot\zeta')\delta(k-k')$$

For $\zeta=\zeta'$: require $\zeta^2 \geq 0$ to avoid ghosts

In summary we require:

$\xi^2 \geq 0$ polarization is light like or space like

$\xi \cdot K = 0$ transverse polarization

$\alpha' K^2 = a - 1$ K is spacelike if $a > 1$
light like if $a = 1$
timelike if $a < 1$

If $a < 1$ K is timelike, $\xi \cdot K = 0 \Rightarrow \xi$ spacelike

$a = 1$ K is null, $\xi \cdot K = 0 \Rightarrow \xi$ spacelike or null

$a > 1$ K is spacelike, $\xi \cdot K = 0 \Rightarrow \xi$ timelike

$\Rightarrow$ ghosts!

Then we reject $a > 1$ ie require $a \leq 1$

However the ground state is a tachyon for $0 < a \leq 1$

Critical theory: $a=1$: $K^2=0$ $(S^2 \geq 0)$

$S \cdot \alpha_{-1} |0, K\rangle = |S, K\rangle$ is a massless state

Consider now the state $|K; K\rangle = K \cdot \alpha_{-1} |0, K\rangle$
Clearly, this state has zero norm (so it is physical) and has "longitudinal" polarization

Moreover, this state is orthogonal to <u>all</u> physical states

$$\langle K; K | S; K' \rangle = (K \cdot S)\, \delta(K-K') = 0$$

So the longitudinal polarization decouples leaving only $D-2$ physical polarizations (like a photon!)

However: for $a=1$ ground state is a tachyon

Note that for the longitudinal degree of freedom the decoupling is due to the fact that $|K;K\rangle$ is a "pure gauge" state in the following sense. Consider

$$L_{-1}|0;K\rangle = e K\cdot\alpha_{-1}|0;K\rangle = e|K;K\rangle$$

ie $|K;K\rangle$ created by the action of L_{-1}.

Recalling that L_{-1} is a generator of a conformal transformation, we say that $|K;K\rangle$ is a pure gauge" state.

$\boxed{3}$ Old covariant quantization

this lecture

- Hilbert space (without constraints)
- Normal ordering
- Virasoro algebra
- Imposing the constraints and the phys
- Mass shell & level-matching conditions
- level 0 & 1 ; dealing with ghosts

Next: more on physical states