STRING THEORY I

Lecture 6

3 Old covariant quantization

last lecture
- Hilbert space (without constraints)
- Normal ordering & Virasoro algebra
- Imposing the constraints and the phys
- Mass shell & level-matching conditions
- level 0 & 1 states dealing with ghosts

this lecture
- the phys, spurious states, null states, ghosts
- critical dimension

Summarize last lecture:

① Hilbert space (of states **before** imposing constraints)

Let

$$\mathcal{H}^{Fock} = \text{Span}\left\{ \prod_{i=1}^{K} \alpha^{\mu}_{-n_i} |0\rangle \right\} = \bigoplus_{N=1}^{\infty} \mathcal{H}^{Fock}[N]$$

Fock space of open string

vacuum annihilated by α^{μ}_{n}, n +ve

creation op

evalue of number op N

Then

$$\mathcal{H}_{open} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{Fock}$$

ground state

$$\mathcal{H}_{closed} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{Fock} \otimes \tilde{\mathcal{H}}^{Fock}$$

right

left $\tilde{\alpha}$

② Virasoro algebra

$$L_m = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{m-k} \cdot \alpha_k \qquad m \neq 0$$

normal ordered

$$L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + N \qquad N = \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k$$

number operator

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}$$

Virasoro algebra with central charge $c = D$

(similarly for $\tilde{L}$)

③ Physical states:

$$\mathcal{H}_{phys} = \left[\bigcap_{m=1}^{\infty} \mathrm{Ker}(L_m) \right] \cap \mathrm{Ker}(L_0 - a)$$

physical states

$\forall m \geq 1 : L_m |\psi\rangle = 0$ & $(L_0 - a)|\psi\rangle = 0$

a: normal ordering constant.

(This page is not in recorded lectures)

Remark:

$$[N, L_m] = -L_m \implies L_m : \mathcal{H}_{Fock}[N] \longrightarrow \mathcal{H}_{Fock}[N-m]$$

(action of L_m is to shift $N \to N-m$)

Moreover: $[L_{m+1}, L_m] = L_{m+2}$

$\implies L_1$ & L_2 generate every L_m $m \geq 3$!

so only need to impose $L_1|\phi\rangle = 0$ & $L_2|\phi\rangle = 0$

So

$$\mathcal{H}_{phys} = \mathrm{Ker}(L_1) \cap \mathrm{Ker}(L_2) \cap \mathrm{Ker}(L_0 - a)$$

④ level 0 & level 1 physical states (open strings)

level zero : $|0;k\rangle$ ground state with $\alpha' k^2 = a$

giving states which are massive for $a<0$ $k^2 = -M^2$

massless for $a=0$

tachyons for $a>0$

level one : $|\xi;k\rangle = (\xi\cdot\alpha_{-1})|0;k\rangle$

general state ↑ polarization $\xi^\mu \in \mathbb{R}^{1,D-1}$

These obey $\alpha' k^2 = a-1$, $\xi\cdot k = 0$ & norm $= \xi^2$

Need to require $a \leq 1$ to avoid ghosts

We studied the threshold case where <u>$a=1$</u> so $K^2=0$:

There is a state with zero norm

$$|K;K\rangle = (K\cdot\alpha_{-1})|0;K\rangle$$

which is a physical ($K^2=0$) "longitudinally polarized" state and, moreover, transverse to all physical states

Hence, the longitudinal polarization <u>decouples</u> and we are left with $D-2$ physical polarizations

Note that
- the ground state for $a=1$ is a tachyon
- $|K;K\rangle$ is created by the action of L_{-1}

$$L_{-1}|0;K\rangle = \ell|K;K\rangle$$

ie $|K;K\rangle$ is "pure gauge"

Definition: a state is called spurious if it is orthogonal to all physical states and obeys

$$(L_0 - a)|\psi\rangle = 0$$

A spurious state which is also physical is orthogonal to itself, ie it has zero norm. These are called null states.

An example of a null state is the state $|K;k\rangle$ at level 1 & $a=1$ as we described.

$|\psi\rangle$ spurious $\Rightarrow$ $\langle \phi | \psi \rangle = 0$ $\quad \forall |\phi\rangle \in \mathcal{H}_{phys}$

If $|\psi\rangle$ is also physical $\Rightarrow$ $\langle \psi | \psi \rangle = 0$

Recall that in gauge theories, a consequence of residual symmetries is that one expects to find states which are pure gauge states. These should be "quotiented out". That is we want to identify

$$|\psi\rangle_{phys} \sim |\psi\rangle_{phys} + |\chi\rangle_{null}$$

So

$$\mathcal{H} = \mathcal{H}_{phys} / \mathcal{H}_{null}$$

In the example at level 1 with $a=1$, null states are created by the action of L_{-1} on the ground state, and can be quotiented.

Consider the state $L_{-1}|0,K\rangle$

The state $L_{-1}|0;K\rangle$ is manifestly perpendicular to physical states.

$\langle\varphi|L_{-1}|0;K\rangle = 0 \qquad |\varphi\rangle \in \mathcal{H}_{phys}$

Now $(L_0 - a)L_{-1}|0;K\rangle = L_{-1}(L_0 - a + 1)|0;K\rangle$

Thus $L_{-1}|0;K\rangle$ is spurious when

$$L_0|0;K\rangle = (a-1)|0;K\rangle$$

$$L_0 L_{-1} = \underbrace{[L_0, L_{-1}]}_{L_{-1}} + L_{-1}L_0 = L_{-1}(L_0+1)$$

The state $L_{-1}|0;k\rangle$ is also physical if $\underline{a=1}$ ↘

($\Rightarrow$ it is null)

Check:

The state $L_{-1}|0;k\rangle$ is physical if $L_m L_{-1}|0;k\rangle = 0 \quad \forall m \geq 1$

$\boxed{L_1}$

$$L_1 L_{-1}|0;k\rangle = (2L_0 + L_{-1}L_1)|0;k\rangle$$

$$= 2L_0|0;k\rangle = 2(a-1)|0;k\rangle = 0 \quad \text{iff } \underline{a=1}$$

$\boxed{L_m}$ $m \geq 2$

$$L_m L_{-1}|0;k\rangle = [(m+1)\underbrace{L_{m-1}}_{m-1 \geq 1} + L_{-1}L_m]|0;k\rangle = 0$$

(but we knew this!)

In fact: all spurious states are of the form

$$|\xi\rangle = \sum_{m=1}^{\infty} L_{-m}|\tilde{\chi}_m\rangle \quad \text{with} \quad L_0|\tilde{\chi}_m\rangle = (a-m)|\tilde{\chi}_m\rangle$$

This series can be equivalently written as

$$|\xi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle \quad \text{with} \quad L_0|\chi_m\rangle = (a-m)|\chi_m\rangle$$

This is because for $m \geq 3$: can replace L_{-m} by commutators

$$[L_{-p}, L_{-q}] = (-p+q)\, L_{-p-q}, \quad 1 < p, q < m$$

eg $\quad [L_{-1}, L_{-2}] = L_{-3}$

$$L_{-3}|\tilde{\chi}_3\rangle = [L_{-1}, L_{-2}]|\tilde{\chi}_3\rangle = L_{-1}(L_{-2}|\tilde{\chi}_3\rangle) - L_{-2}(L_{-1}|\tilde{\chi}_3\rangle)$$

$$= L_{-1}|\chi_1'\rangle + L_{-2}|\chi_2'\rangle$$

etc. ... (exercise)

Check: mostly an exercise

$|\xi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle \qquad L_0|\chi_m\rangle = (a-m)|\chi_m\rangle$

- clearly $\langle\varphi|\xi\rangle = 0 \quad \forall \; |\varphi\rangle \in \mathcal{H}_{phys}$
- $(L_0 - a)|\xi\rangle = 0$

so $|\xi\rangle$ is spurious. To see that all spurious states have this form see GSW p 83

Constructing null states

Consider $|\xi\rangle = L_{-1}|\chi\rangle$ s.t $L_m|\chi\rangle = 0 \quad \forall m>0$

$L_0|\chi\rangle = (a-1)|\chi\rangle$

Then: check $L_m|\xi\rangle = 0 \quad \forall m\geq 1$

$$L_{+1}|\xi\rangle = \left([L_{+1}, L_{-1}] + L_{-1}L_{+1}\right)|\chi\rangle$$

$$= 2L_0|\chi\rangle = 2(a-1)|\chi\rangle = 0 \quad \text{iff } a=1$$

$L_m|\xi\rangle = 0 \quad \forall m\geq 2$

eg $L_{+2}|\xi\rangle = \left([L_{+2}, L_{-1}] + L_{-1}L_{+2}\right)|\chi\rangle = 3L_{+1}|\chi\rangle = 0$

So for $a=1$ we get an infinite set of null states.
The example we had earlier $|k;k\rangle \sim L_{-1}|0;k\rangle$ is the simplest case in this family.

Consider now, the spurious state

$$|\xi\rangle = (L_{-2} + \gamma L_{-1}^2)|\chi\rangle \quad \text{with} \quad L_m|\chi\rangle = 0,\ m>0$$
$$L_0|\chi\rangle = (a-2)|\chi\rangle$$

Then $L_m|\xi\rangle = 0 \quad \forall\ m \geq 3$

$$L_1|\xi\rangle = \left([L_1, L_{-2}] + L_{-2}L_{+1} + \gamma([L_1, L_{-1}] + L_{-1}L_1)L_{-1}\right)|\chi\rangle$$
$$= \left(3L_{-1} + 2\gamma L_0 L_{-1} + \gamma L_{-1}([L_1, L_{-1}] + L_{-1}L_1)\right)|\chi\rangle$$
$$= (3L_{-1} + 2\gamma([L_0, L_{-1}] + L_{-1}L_0) + 2\gamma L_{-1}L_0)|\chi\rangle$$

($[L_0, L_{-1}] = +L_{-1}$)

$$= ((3+2\gamma)L_{-1} + 4\gamma(a-2)L_{-1})|\chi\rangle$$
$$= (3 + 2\gamma(2a-3))L_{-1}|\chi\rangle$$

$L_1|\xi\rangle = 0$ iff $\gamma = \frac{3}{2(3-2a)}$ $\quad \left(\gamma = \frac{3}{2},\ a=1\right)$

$$L_{+2}|\xi\rangle = L_{+2}(L_{-2} + \gamma L_{-1}^2)|\chi\rangle$$

$$= \{ [L_{+2}, L_{-2}] + \cancel{L_{-2} L_{+2}} + \gamma([L_{+2}, L_{-1}] + L_{-1} L_{+2}) L_{-1} \} |\chi\rangle$$

$$= \{ 4 L_0 + \frac{D}{12} \cdot 6 + 3\gamma L_{+1} L_{-1} + \gamma \cancel{L_{-1} \cdot 3 L_{+1}} \} |\chi\rangle$$

$$= \{ 4 L_0 + \frac{1}{2} D + 3\gamma \cdot 2 L_0 \} |\chi\rangle$$

$$= \{ 2(2 + 3\gamma)(a-2) + \frac{1}{2} D \} |\chi\rangle$$

$L_{+2}|\xi\rangle = 0$ iff $D = 4(2-a)(2+3\gamma)$

For $\underline{a=1}$: $\underline{D=26}$ critical bosonic string

$\uparrow$ critical dim

there are **more** null states of the form

$$(L_{-2} + \frac{3}{2} L_{-1}^2)|\chi\rangle, \quad L_m|\chi\rangle = 0 \ m>0, \ (L_0+1)|\chi\rangle = 0$$

Let $a = 1$:

This is the threshold value as for $a > 1$ there are ghosts in the phys.

Consider the level-2 state $|\phi\rangle$ with momentum K

$$|\phi, K\rangle = [c_1\, \alpha_{-1} \cdot \alpha_{-1} + c_2\, \alpha_{-2} \cdot \alpha_0 + c_3\, (\alpha_{-1} \cdot \alpha_0)(\alpha_{-1} \cdot \alpha_0)]\, |0; K\rangle$$

onshell mass condition $-\alpha' K^2 = 1$ $\quad (\alpha' M^2 = N - a)$

$[L_m, \alpha_n^\mu] = -n\, \alpha_{m+n}^\mu$

$L_1 |\phi, K\rangle = 0$ iff $c_1 + c_2 - 2c_3 = 0$

$L_2 |\phi, K\rangle = 0$ iff $Dc_1 - 4c_2 - 2c_3 = 0$

so $|\phi, K\rangle = [\alpha_1 \cdot \alpha_1 + \frac{1}{5}(D-1)\, \alpha_{-2} \cdot \alpha_0 + \frac{1}{10}(D+4)(\alpha_{-1} \cdot \alpha_0)(\alpha_{-1} \cdot \alpha_0)]\, |0, K\rangle$

is a physical state for any D $\quad (c_1 = 1)$

Norm $\langle \phi; k | \phi; k' \rangle = \frac{2}{25}(D-1)(D-26)\,\delta(k-k')$

- no ghosts : $1 \leq D \leq 26$
- null state when $D=1$ or $D=26$

Therefore when $a=1$ & $D=26$ there are **more** null states

1972 Brower ; Goddard, Thorn No ghost theorem
↳ For $a=1$ and $D=26$ the phys has no ghosts

Note: there are no ghosts for $a \leq 1$ & $1 \leq D \leq 25$
but these theories are inconsistent at the level of string loops (need to look at one loop interactions)

OCQ: $a=1$ $D=26$ needs 1-loop interactions to prove

(no ghosts, many null states)

LCQ: • $a=1$ $D=26$ follows by requiring Lorentz spacetime invariance

• manifestly ghost free

BRST quantization: $a=1$, $D=26$ required for quantum gauge (conformal) invariance.