STRING THEORY 1

Lecture 7

3 OLD COVARIANT QUANTIZATION

so far

- Hilbert space (without constraints)
- Virasoro algebra
- Imposing constraints and the physical states
- Mass-shell & level matching conditions
- Level 0 & 1 states, dealing with ghosts
- Spurious states, null states
- critical dimension

Today: • comments on the no ghost theorem for subcritical strings, ($a \leq 1$, $D \leq 25$)

• closed string spectrum

Open strings: by studying the low level spectrum we found

- $a > 1$ $D > 26$: there are ghosts in the $\mathcal{H}_{phys}$ $\boxed{\Rightarrow a \leq 1}$
- $a = 1$ $D = 26$ critical string ⟵

lots of null states — we found two infinite families of null states

- $a \leq 1$ $D \leq 25$ (subcritical case)

no inconsistencies at tree-level of the free spectrum (no ghosts)
but inconsistent at the level of string loops (need to look at one loop interactions)

[Recall that to construct $\mathcal{H}_{phys}$ we have only imposed half the constraints ($L_m|\phi\rangle = 0$ $m \geq 1$ $(L_0 - a)|\phi\rangle = 0$)
The remaining charges L_{-m} $(m>0)$ give redundancies associated to residual gauge symmetries and we consider
$$|\psi\rangle \sim |\psi\rangle + |\psi\rangle_{null} \quad (\mathcal{H}_{phys}/\mathcal{H}_{null})]$$

Subcritical string states (no ghosts for $1 \leq D \leq 25$, $a \leq 1$)

(Starting with the critical string $a=1$ $D=26$)

Consider the states with

$$K = (K^0, K^1, K^2, \dots, K^{24}, \xi) \qquad \text{fixed } \xi \in \mathbb{R}$$

$$N_{25} \equiv \sum_{n>0} \alpha^{25}_{-n} \alpha^{25}_{n} \qquad \text{so no } X^{25} \text{ oscillators activated}$$

The mass-shell condition for these states is

$$\alpha' K^2 = a - N \implies \alpha'(K_i \cdot K^i + \xi^2) = \overset{\leftarrow a=1}{1} - N = 1 - N_{0 \dots 24}$$

$i = 0, 1, \dots, 24$ number operator with N_{25} omitted

$$\implies \alpha' K_i K^i = \boxed{(1 - \alpha' \xi^2)} - N_{0,24}$$

which can be identified to $D=25$ mass shell condition for $a = 1 - \alpha' \xi^2 \leq 1$

$\implies$ no ghost theorem for the critical string implies the no ghost theorem for the subcritical string.

Physical states of the closed string (at least low level $N=0,1$)

Recall $$\mathcal{H}_{\text{closed}} = L^2(\mathbb{R}^{1,D-1}) \otimes \underbrace{\mathcal{H}_{\text{Fock}}}_{\text{left movers}} \otimes \underbrace{\tilde{\mathcal{H}}_{\text{Fock}}}_{\text{right movers}}$$

(states are of the form $\prod_{i=1}^{K} \alpha_{-n_i}^{\mu_i} \prod_{j=1}^{K} \tilde{\alpha}_{-m_j}^{\nu_j} |0;K\rangle$, $n_i, m_j \geq 1$)

Physical state conditions:

$$(L_0 + \tilde{L}_0 - 2a)|\phi\rangle = 0 \iff \alpha' K^2 = 4a - 2(N+\tilde{N}) \overset{a=1}{=} 2(2-N-\tilde{N})$$

$$(L_0 - \tilde{L}_0)|\phi\rangle = 0 \iff N = \tilde{N}$$

$$L_m|\phi\rangle = 0 \quad \& \quad \tilde{L}_m|\phi\rangle \quad \forall\, m \geq 1$$

sufficient to prove this for $\underline{m=1,2}$

(see page 6 lecture 6)

level 0

$N = \tilde{N} = 0$

ground state $|0;k\rangle$, $\qquad \alpha' k^2 = 4a \overset{a=1}{=} 4$ tachyon

Level 1

$N = \tilde{N} = 1$

$|\Omega; k\rangle \equiv \Omega_{\mu\nu}\, \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |0;k\rangle, \qquad \alpha' k^2 = 4a - 4 \overset{a=1}{=} 0$

spacetime two tensor

massless state for the critical string

Decompose this state into space-time Lorentz irreps

$$\Omega_{\mu\nu} = \gamma_{\mu\nu} + \varphi\, \eta_{\mu\nu} + b_{\mu\nu}$$

traceless symmetric $\qquad$ trace $\qquad$ antisymmetric

$$D^2 = \tfrac{1}{2} D(D+1) - 1 \; + \; 1 \; + \; \tfrac{1}{2} D(D-1)$$

Impose Virasoro constraints:

$$L_m |\Omega, K\rangle = 0 \quad \& \quad \tilde{L}_m |\Omega, K\rangle = 0 \qquad m \geq 1$$

It is sufficient to show this for $m=1$

$$L_1 |\Omega; K\rangle = \Omega_{\mu\nu} L_1 \alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1} |0; K\rangle$$

$$= \Omega_{\mu\nu} (\alpha^\mu_0 \tilde{\alpha}^\nu_{-1}) |0; K\rangle$$

$$= \frac{\ell}{2} K^\mu \Omega_{\mu\nu} \tilde{\alpha}^\nu_{-1} |0; K\rangle$$

similarly

$$\tilde{L}_1 |\Omega; K\rangle = \frac{\ell}{2} K^\nu \Omega_{\mu\nu} \alpha^\mu_{-1} |0; K\rangle$$

For γ

$$\frac{\ell}{2} K^\mu \gamma_{\mu\nu} \tilde{\alpha}_{-1}^\nu |0; K\rangle = 0 \quad \text{if} \quad K^\mu \gamma_{\mu\nu} = 0$$
$$\frac{\ell}{2} K^\nu \gamma_{\mu\nu} \alpha_{-1}^\mu |0; K\rangle = 0 \quad \text{if} \quad K^\nu \gamma_{\mu\nu} = 0$$

$\gamma_{\mu\nu}$ polarization is transverse traceless

$$\left(\frac{1}{2} D(D+1) - 1\right) - D = \frac{1}{2} D(D-1) - 1$$

For b

$$\frac{\ell}{2} K^\mu b_{\mu\nu} \tilde{\alpha}_{-1}^\nu |0; K\rangle = 0 \quad \text{if} \quad K^\mu b_{\mu\nu} = 0$$
$$\frac{\ell}{2} K^\nu b_{\mu\nu} \alpha_{-1}^\mu |0; K\rangle = 0 \quad \text{if} \quad K^\nu b_{\mu\nu} = 0$$

$b_{\mu\nu}$ polarization is transverse antisymmetric

$$\frac{1}{2} D(D-1) - D = \frac{1}{2} D(D-3)$$

For φ

$$\frac{\ell}{2} \varphi K \cdot \tilde{\alpha}_{-1} |0; K\rangle = 0 \quad \text{if} \quad K = 0 \ ?$$
$$\frac{\ell}{2} \varphi K \cdot \alpha_{-1} |0; K\rangle = 0 \quad \text{if} \quad K = 0 \ ?$$

unphysical constraint

↳ see later

As for the open string, there are redundancies in the phys associated to the fact that we only impose $L_m|\psi\rangle = 0 \quad m \geq 1$.
That is, there is some remaining gauge invariance associated to null states.

In fact: $L_{-1}\left((\xi \cdot \tilde{\alpha}_{-1})|0;K\rangle\right) = \frac{\ell}{2} K_\mu \xi_\nu \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0;K\rangle$

so states with $\quad \Omega_{\mu\nu} = K_\mu \xi_\nu \quad$ are spurious

They are also physical (so null) if $\quad K^\mu (K_\mu \xi_\nu) = 0 \quad \& \quad K^\nu (K_\mu \xi_\nu)$

that is, if $\quad K^2 = 0 \quad \& \quad K \cdot \xi = 0$

[$L_{-1}(\xi \cdot \tilde{\alpha}_{-1})|0;K\rangle = (\xi \cdot \tilde{\alpha}_{-1}) L_{-1}|0;K\rangle$

$L_{-1}|0;K\rangle = \frac{1}{2} \sum_{n \in \mathbb{Z}} (\alpha_{-1-n} \cdot \alpha_n)|0;K\rangle \qquad$ n-terms vanishes for $n \geq 1$ or $n \leq -2$

$= \frac{1}{2} [\alpha_{-1} \cdot \alpha_0 + \alpha_0 \cdot \alpha_{-1}]|0;K\rangle = \frac{\ell}{2} (K \cdot \alpha_{-1})|0;K\rangle \qquad \alpha_0^\mu = \frac{\ell}{2} p^\mu$]

Then we have the invariances

$$\gamma_{\mu\nu} \longrightarrow \gamma_{\mu\nu} + k_\mu k_\nu + \xi_\nu k_\mu \qquad \xi \cdot k = 0$$

$$b_{\mu\nu} \longrightarrow b_{\mu\nu} + \xi_\mu k_\nu - \xi_\nu k_\mu \qquad (\xi_\mu \sim \xi_\mu + k_\mu \phi)$$

Interpretation

- traceless symmetric state $\rightsquigarrow$ graviton

ie describes a massless transv. polarized spin 2 particle

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x)$$

ξ = infinitesimal coord transf
$X \to X + \xi$

where $\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu}(x) + \boxed{\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x)}$

which in momentum space $\gamma_{\mu\nu}(k) \sim \gamma_{\mu\nu}(k) + k_\mu \xi_\nu + k_\nu \xi_\mu$ with $k \cdot \xi = 0$

$$\underbrace{\tfrac{1}{2}D(D+1)}_{\text{symmetric 2-tensor}} - \underbrace{1}_{\text{trace}} - \underbrace{D}_{k^\mu \gamma_{\mu\nu} = 0} - \underbrace{(D-1)}_{k \cdot \xi = 0} = \tfrac{1}{2}(D-2)(D-1) - 1$$

dim of irrep of SO(D-2) for massless transv. polarized spin 2 particle

• anti-symmetric state ⇝ $\boxed{\text{2 form}}$ gauge field
(Kalb-Ramond field)

$$b^{(2)} = \frac{1}{2} b_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu \sim b^{(2)} + da^{(1)} \qquad a^{(1)} \sim a^{(1)} + d\lambda$$

in momentum space $\quad b_{\mu\nu}(K) \sim b_{\mu\nu}(K) + K_\mu \xi_\nu - K_\nu \xi_\mu$

with $\xi_\mu \sim \xi_\mu + K_\mu \phi$

$$\frac{1}{2} D(D-1) - (D-1) - (D-2) = \frac{1}{2}(D-2)(D-3)$$

SO(D-2) deg of freedom for massless antisymmetric two-form field

- How about the scalar? We have $S_{\mu\nu} = \frac{1}{2}\varphi\eta_{\mu\nu}$

We have the state $\frac{1}{2}\varphi\, \alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;k\rangle$

However

$$L_1(\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;k\rangle) = \alpha_0\cdot\tilde{\alpha}_{-1}|0;k\rangle = \frac{\ell}{2}k\cdot\tilde{\alpha}_{-1}|0;k\rangle \neq 0$$

(and similarly for $\tilde{L}_1$)

Then this is not a physical state!

However we can construct a physical level one state which is a space-time scalar

Define the state

$$|\varphi_{\xi,\tilde{\xi}};k\rangle = \varphi\left[(\xi\cdot\alpha_{-1})\left(\frac{\ell k}{2}\cdot\tilde{\alpha}_{-1}\right) + \left(\frac{\ell k}{2}\cdot\alpha_{-1}\right)(\tilde{\xi}\cdot\tilde{\alpha}_{-1}) + \alpha_{-1}\cdot\tilde{\alpha}_{-1}\right]|0;k\rangle$$

Now impose the Virasoro constraints

Virasoro constraints:

$$L_1 |\varphi_{\zeta,\tilde\zeta};k\rangle = \varphi\Big[\zeta\cdot(\alpha_0 + \cancel{\alpha_{-1}L_1})\left(\frac{\ell k}{2}\cdot\tilde\alpha_{-1}\right)$$

$$+ \frac{\ell k}{2}(\alpha_0 + \cancel{\alpha_{-1}L_1})(\tilde\zeta\cdot\tilde\alpha_{-1}) + (\alpha_0 + \cancel{\alpha_{-1}L_1})\cdot\tilde\alpha_{-1}\Big]|0,k\rangle$$

$$= \varphi\frac{\ell}{2}\Big[(\zeta\cdot k)\left(\ell\frac{k}{2}\cdot\tilde\alpha_{-1}\right) + \frac{\ell}{2}\underbrace{k^2}_{0}(\tilde\zeta\cdot\tilde\alpha_{-1}) + k\cdot\tilde\alpha_{-1}\Big]|0,k\rangle$$

$$= \varphi\frac{\ell}{2}\underbrace{\left(\frac{\ell}{2}(\zeta\cdot k)+1\right)}(k\cdot\tilde\alpha_{-1})|0,k\rangle$$

$= 0$ iff $\zeta\cdot k = -\frac{2}{\ell}$

$$\tilde L_1 |\varphi_{\zeta,\tilde\zeta};k\rangle = \varphi\frac{\ell}{2}\left(\frac{\ell}{2}(\tilde\zeta\cdot k)+1\right)(k\cdot\alpha_{-1})|0,k\rangle$$

$= 0$ iff $\tilde\zeta\cdot k = -\frac{2}{\ell}$

So $|\varphi_{\zeta,\tilde\zeta};k\rangle$ is a physical state when $\zeta\cdot k = -2/\ell$, $\tilde\zeta\cdot k = -\frac{2}{\ell}$

This state corresponds to a scalar despite the fact that it seems to depend on ς & $\tilde{\varsigma}$!

To see this, consider how the state depends on the choice of ξ & $\tilde{\xi}$

$$|\varphi_{\varsigma,\tilde{\varsigma}};K\rangle - |\varphi_{\varsigma',\tilde{\varsigma}};K\rangle$$

(and similarly for $\tilde{\xi}$ & $\tilde{\xi}'$)

$$= \varphi\,\big((\varsigma-\varsigma')\cdot\alpha_{-1}\big)\Big(\frac{\ell K}{2}\cdot\tilde{\alpha}_{-1}\Big)|0;K\rangle$$

$$= \varphi\,\tilde{L}_{-1}\big((\varsigma-\varsigma')\cdot\alpha_{-1}\big)|0;K\rangle$$

So setting $\lambda = \varsigma - \varsigma'$

$$\tilde{L}_{-1}(\lambda\cdot\alpha_{-1})|0;K\rangle = (\lambda\cdot\alpha_{-1})(\tilde{\alpha}_0\cdot\tilde{\alpha}_{-1})|0;K\rangle$$

$$= (\lambda\cdot\alpha_{-1})\Big(\frac{\ell K}{2}\cdot\tilde{\alpha}_{-1}\Big)|0;K\rangle$$

(and similarly for $\tilde{\xi}$ & $\tilde{\xi}'$)

Then

$$|\varphi_{s,\tilde{s}};K\rangle - |\varphi_{s',\tilde{s}};K\rangle$$
$$|\varphi_{s,\tilde{s}};K\rangle - |\varphi_{s,\tilde{s}'};K\rangle$$
are spurious

Moreover, they are null

$$(s-s')\cdot K = \underbrace{s\cdot K}_{-\ell/2} - \underbrace{s'\cdot K}_{-\ell/2} = 0 \; ; \qquad (\tilde{s}-\tilde{s}')\cdot K = 0$$

Hence we identify a state with s ($\bar{s}$) an s' ($\bar{s}'$)

$$|\varphi_{s,\tilde{s}};K\rangle \sim |\varphi_{s',\tilde{s}};K\rangle + \varphi\,\tilde{L}_{-1}\left((s-s')\cdot\alpha_{-1}\right)|0;K\rangle$$
$$|\varphi_{s,\tilde{s}};K\rangle \sim |\varphi_{s,\tilde{s}'};K\rangle + \varphi\,L_{-1}\left((\tilde{s}-\tilde{s}')\cdot\tilde{\alpha}_{-1}\right)|0;K\rangle$$

This scalar φ is called the dilaton:
it plays a very important role in the context of string interactions.

Appendix on (a very brief discussion of) the light-cone quantization (see GSW)

The fastest way to construct the phys / null of the quantized string is to fix the remaining gauge freedom by choosing the light-cone gauge.

One defines
$$X^{\pm}(\tau,\sigma) = \frac{1}{2}\left(X^0 \pm X^{D-1}\right)$$

and uses the remaining gauge freedom to set
$$X^+ = x^+ + p^+\tau \qquad \text{no oscillators !}$$

Then one uses the Virasoro constraints for X^-.

The result is that the space of physical states is constructed from

$D-2$ spacelike transverse oscillators α_n^i $\quad i=1,\ldots,D-2$

(and $\tilde{\alpha}_n^i$ for the closed string)

- $\alpha' M^2 = (N^\perp - 1)$ $\quad$ (and $\alpha' M^2 = (N^\perp + \tilde{N}^\perp - 2)$ $\quad a=1$

for the closed string)

where $N^\perp = \sum_{n=1}^{\infty} \delta_{ij} \alpha_{-n}^i \alpha_n^j$, transverse number operator

- There are no negative norm states

Counting states is very fast: some examples

open string at level 1 $\quad \alpha^i_{-1}|0;k\rangle \quad i=1,\dots,D-2$

D-2 states transforming as vector of SO(D-2)
which only makes sense if this is massless
(for which the little group is SO(D-2))
[For a massive state need an extra state as these fall into reps of SO(D-1)]

This implies $a=1$ for Lorentz invariance

more detailed calculation $\Rightarrow$ Also one needs $D=26$ (see GSW)

open string level 2 (see problem sheet)

The states $\alpha^i_{-1}\alpha^j_{-1}|0;k\rangle$, $\alpha^i_{-2}|0;k\rangle$

sym $D-1$ indices

give $\frac{1}{2}(D-1)(D-2) + D-2 = \frac{1}{2}(D+1)(D-2)$ states

$$= \frac{1}{2}D(D-1) - 1$$

symmetric 2-tensor of $SO(D-1)$ trace

this is interpreted as a **massive** spin-2 (which under the full $SO(1,D-1)$ corresponds to a transverse, traceless symmetric 2 tensor)

Final remark:

We have gone over the OCQ quantization of the relativistic string

This is a first quantization meaning that the states are one particle states.

Second quantization has to do with string field theory (not well understood!)

Too bad we have no time to go over the modern (BRT) quant.! Instead learn about D-branes, T-duality etc,

<u>Next</u> : vertex operators, interactions.