STRING THEORY I

Lecture 8

4 Interactions

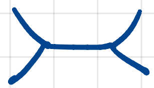
4.1 Generalities

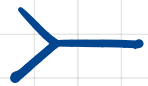
QFT:

- to understand interactions one adds non-linear terms to the action

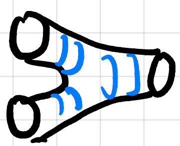
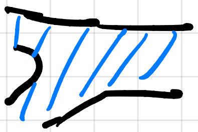
  ↳ doesn't work for the string because anything you try to add breaks gauge invariance.

- scattering amplitudes → Feynman diagrams

eg  etc

interactions encoded at vertices say 

↳ in string theory this is replaced by, for instance

 or  : no such vertices!

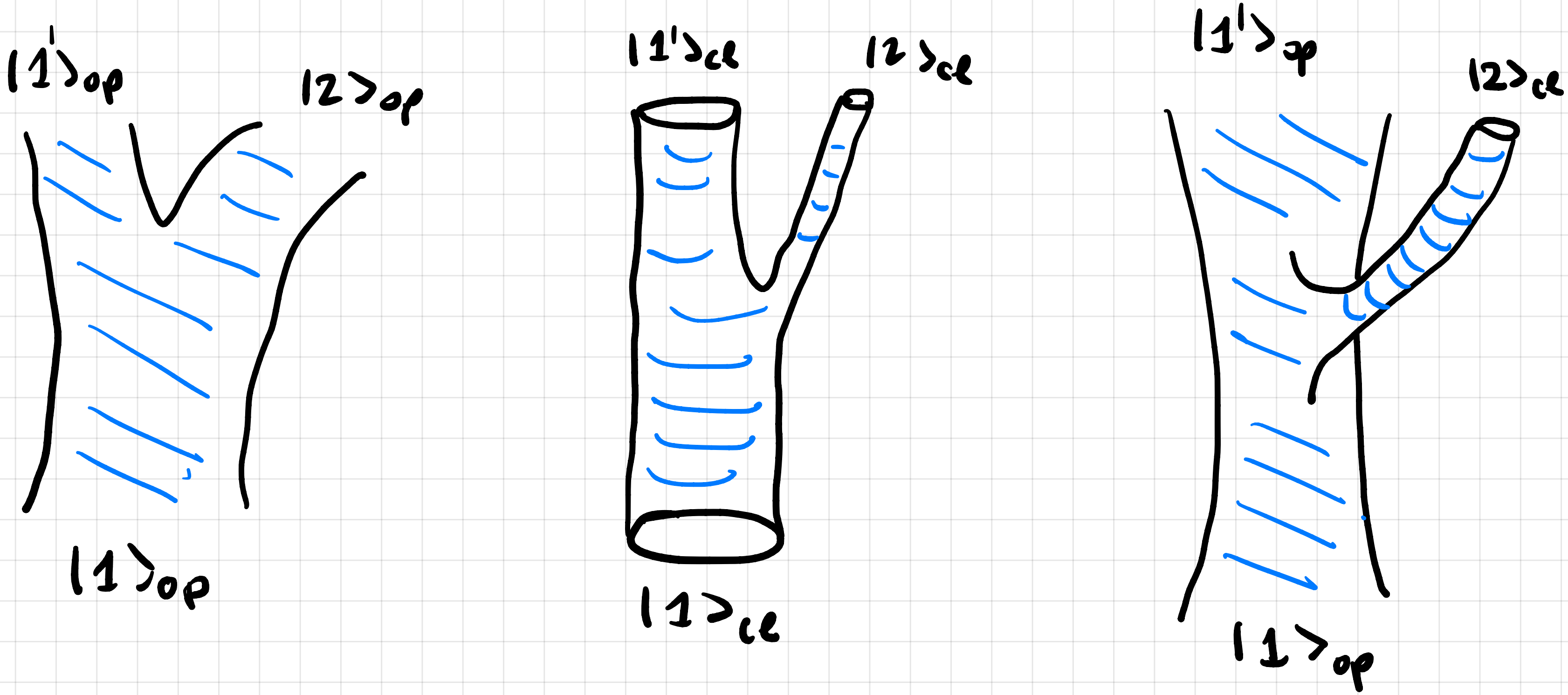
In string theory:

Want to compute for example the amplitude of a given configuration of quantized strings at an initial time to evolve into a new configuration at a later time

Problem: it is not known how to do this ....

we need to work with the first quantized formalism

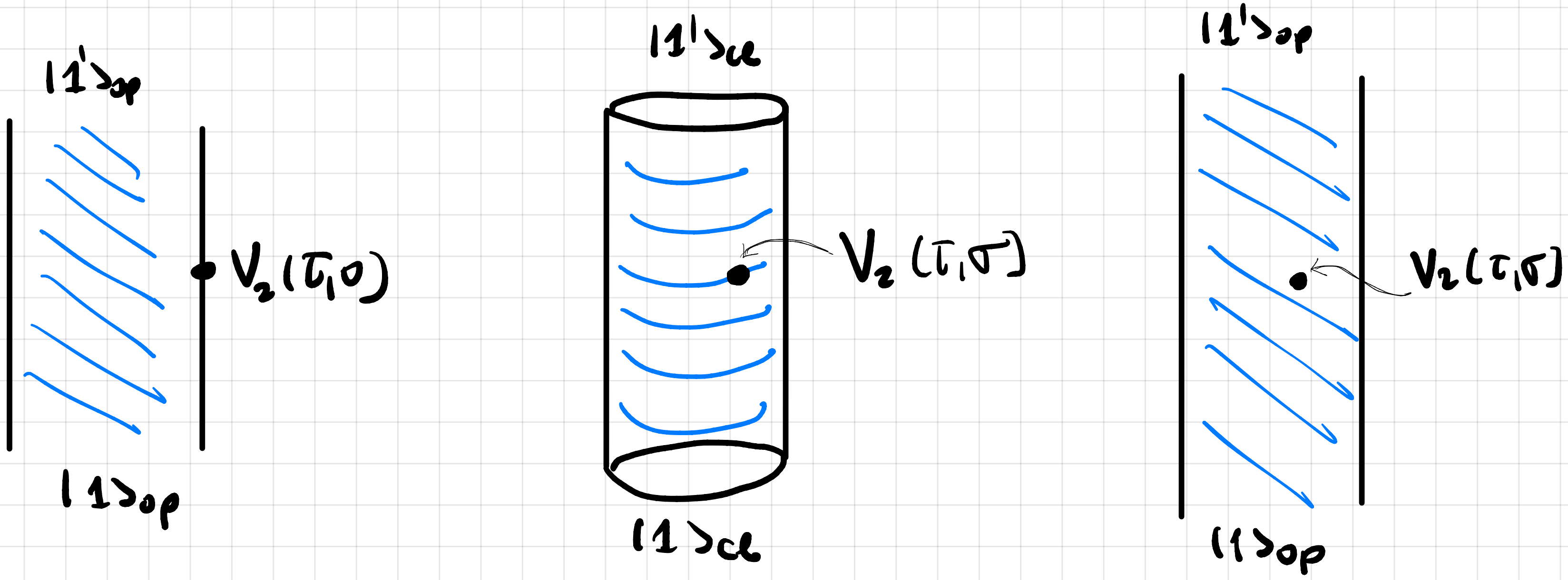
We start by considering processes in which a string breaks into two (or two join to give a single string)



How?

Suppose $|2\rangle$ is a physical state which is emitted / absorbed.

We describe emission or absorption of a given quantum state (say $|2\rangle$) from a string world sheet by the action of a local operator or Vertex operator



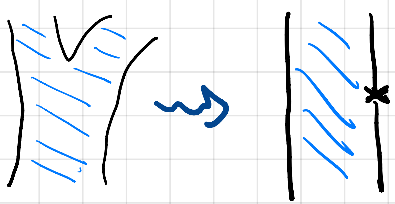
One explanation for this is that $|2\rangle$, as a physical state emitted or absorbed, is a <u>quantum</u> state with "mass squared and width

GS of order $\hbar$". Arguably in the classical limit these states behave like point particles.

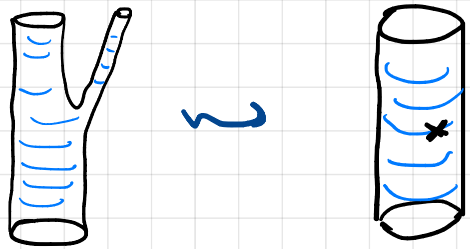
<u>Instead</u>: (Chapter 1 GSW)

- Wick rotation of the world sheet (Lorentzian signature → Euclidean signature)
- conformal transformation

accept this!



or

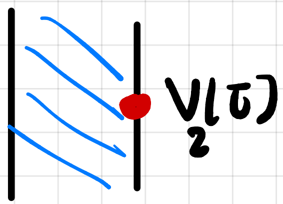


4.2 Vertex operators : Introduction

Require:

- time evolution on the worldsheet is a gauge transformation ⇒ position of the vertex operator should not be meaningful.

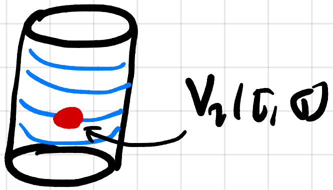
open string vertex operator



$$\int d\tau \underbrace{V_2(\tau)}_{\downarrow}$$

inserted on the boundary

closed string vertex operator



$$\int d\tau\, d\sigma \underbrace{V_2(\tau,\sigma)}_{\downarrow}$$

inserted on the interior

- absorption & emission string operators to map

$$\mathcal{H}_{phys} \longrightarrow \mathcal{H}_{phys}$$

and

$$\mathcal{H}_{null} \longrightarrow \mathcal{H}_{null}$$

Let's see how this works for the open strings first consider the action of the vertex operator on a state $|\phi\rangle$

$$\int d\tau \; V(\tau, 0) \; |\phi\rangle$$

physical state conditions:

$$(L_m - a\,\delta_{m,0}) \left[\int d\tau \; V(\tau, 0) \, |\phi_{phys}\rangle \right] \qquad m \geq 0$$

$$= \int d\tau \; [L_m, V(\tau, 0)] \, |\phi_{phys}\rangle$$

$= 0$ physical if this vanishes

if $\underline{[L_m, V(\tau, 0)] = \partial_\tau(\text{local operator})}$

null states:

$$\int d\bar{\tau}\; V(\bar{\tau},0)\left[L_{-m}|\phi\rangle\right] \qquad m \geq 1$$

$$= \underbrace{\int d\bar{\tau}\left\{[V(\bar{\tau},0), L_{-m}]|\phi\rangle\right.}_{\text{need this to vanish up to a total derivative}} + \left.\underbrace{L_{-m}V(\bar{\tau},0)|\phi\rangle}_{\text{null state}}\right\}$$

$\langle\phi_{\text{phys}}|L_{-m}V(\bar{\tau},0)|\phi\rangle = 0$

This is null if $\underline{[V(\bar{\tau},0), L_{-m}] \sim \partial_{\bar{\tau}}(\text{local op})}$

Thus $[L_m, V(\bar{\tau},0)] \sim \partial_{\bar{\tau}}(\cdots)$

Moreover, conformal transformations of the open string of the form $\tau \to \tilde{\tau}(\tau)$.

We want

$$\int V(\tau, 0)\, d\tau \longrightarrow \int \tilde{V}(\tilde{\tau}, 0)\, d\tilde{\tau} = \int V(\tau, 0)\, d\tau$$

$$= \int V(\tau, 0) \frac{d\tau}{d\tilde{\tau}}\, d\tilde{\tau}$$

for $\int V(\tau, 0)\, d\tau$ to be (invariant) under $\tau \to \tilde{\tau}$

ie $\quad V(\tau, 0) \longrightarrow \tilde{V}(\tilde{\tau}, 0) = V(\tau, 0) \dfrac{d\tau}{d\tilde{\tau}}$

Definition: an operator $A(\tau)$ is a primary operator of weight h if under the transformation

$$\tau \to \tilde{\tau}(\tau)$$

it transforms as

$$A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(\frac{d\tau}{d\tilde{\tau}}\right)^{h}$$

For an operator with $h=1$

$$\int \tilde{A}(\tilde{\tau})\, d\tilde{\tau} = \int A(\tau)\frac{d\tau}{d\tilde{\tau}}\, d\tilde{\tau} = \int A(\tau)\, d\tau$$

ie the integrated operator is invariant.

For infinitesimal transformations

$$\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau)$$

we have

$$A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(1 + h\,\frac{d\epsilon}{d\tau}\right)$$

OTOH

$$\tilde{A}(\tilde{\tau}) = \tilde{A}(\tau+\epsilon) = \tilde{A}(\tau) + \epsilon\,\partial_\tau A(\tau) + \mathcal{O}(\epsilon^2)$$

Then we find the variation of A

$$\delta A(\tau) = \tilde{A}(\tau) - A(\tau) = -\,\epsilon\,\partial_\tau A - h(\partial_\tau \epsilon)A$$

$$= -\,\partial_\tau(\epsilon A) - (h-1)\,\partial_\tau \epsilon\, A$$

[which is a total derivative when $h=1$]

The Virasoro operators generate the transformations

$$\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau) \quad \text{with} \quad \epsilon = i e^{im\tau}$$

Then

$$\delta A(\tau) = e^{im\tau}\left(-i\partial_\tau A + h m A\right)$$

so the action of the Virasoro operators is

$$\boxed{[L_m, A(\tau)] = e^{im\tau}(-i\partial_\tau + mh) A(\tau)}$$

Equivalently, this is the condition for A to have conformal weight h.

$h=1$ $\quad [L_m, A(\tau)] = \partial_\tau(-i e^{im\tau} A(\tau))$ ✓

We need to identify primaries of weight 1 which correspond to the physical states in the string Hilbert space. We use this to compute string amplitudes.

Closed strings:

A primary operator of dimension $(h,\tilde h)$ is an operator transforming under conformal transformations as

$$\phi(\sigma_+,\sigma_-) \longrightarrow \tilde\phi(\tilde\sigma_+,\tilde\sigma_-) = \left(\frac{d\sigma_+}{d\tilde\sigma_+}\right)^{\tilde h}\left(\frac{d\sigma_-}{d\tilde\sigma_-}\right)^{h}\phi(\sigma_+,\sigma_-)$$

The corresponding infinitesimal transformations are

$$\delta\phi(\tau,\sigma) = -\partial_+(\tilde\epsilon\phi) - (\tilde h-1)(\partial_+\tilde\epsilon)\phi - \partial_-(\epsilon\phi) - (h-1)(\partial_-\epsilon)\phi$$

(This is a total derivative if $h=\tilde h=1$)

For $\epsilon = \frac{i}{2}e^{2im\sigma_+}$ this gives the action of L_m:

$$[L_m,\phi(\sigma_\pm)] = \frac{1}{2}e^{2im\sigma_+}\left(-i\partial_+ + 2mh\right)\phi(\sigma_\pm)$$

$$[\tilde L_m,\phi(\sigma_\pm)] = \frac{1}{2}e^{2im\sigma_-}\left(-i\partial_- + 2m\tilde h\right)\phi(\sigma_\pm)$$

$\epsilon = \frac{i}{2}e^{2im\sigma_-}$, $\tilde L_m$

4.3 Vertex operators for the open string

Consider the boundary scalar operator $X^\mu(\tau, 0)$:

$(l = 1$
$(\alpha' = 1/2)$

$$X^\mu(\tau,0) = \underbrace{x^\mu + \tau p^\mu}_{x(\tau)\text{ center of mass}} + i \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau}$$

We can check conformal transformations

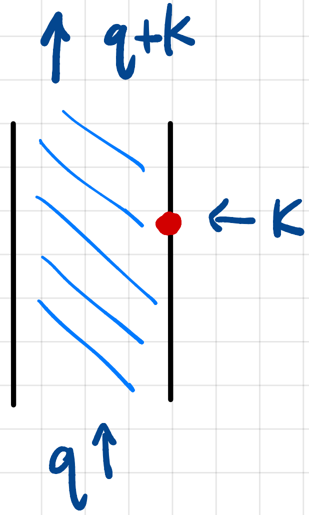
$$L_m = \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n$$

By direct computation

$$[L_m, X^\mu(\tau,0)] = -i \sum_n \alpha_n^\mu e^{-i(n-m)\tau} = -i\left(\frac{d}{d\tau} X^\mu(\tau,0)\right) e^{im\tau}$$

so $h = 0$ ✓

Building a vertex operator associated to a tachyon of momentum k: (level 0, ground state)



so

$$V_k(\tau)\,|0;q\rangle \sim |\phi;q+k\rangle$$

Thus we should demand

$$[P^\mu, V_k(\tau)] = k^\mu V_k(\tau)$$

action of momentum operator on V

This means $V_k(\tau)$ must depend on $e^{ik\cdot x(\tau)}$

A naive guess for $V_k(\tau)$ is

$$V_k(\tau) = e^{ik\cdot X(\tau)}$$

As it stands, it is not well defined: it still needs normal ordering.

Consider the normal ordered expression

$$:e^{ik\cdot X(\tau)}: = e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-m}}{m}e^{im\tau}}\, e^{ik\cdot x(\tau)}\, e^{-k\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}$$

Remark:

$$k_\mu k_\nu [\alpha_m^\mu, \alpha_n^\nu]$$

$$e^{ik\cdot\alpha_m}\, e^{ik\cdot\alpha_n} = e^{ik\cdot(\alpha_m+\alpha_n)}\, e^{-\frac{1}{2}\boxed{k\cdot k\, m\,\delta_{m+n,0}}}$$

so reordering is free if $k^2=0$

Next, we need to compute the conformal dimension

$$[L_m, :e^{ik\cdot X(\tau)}:] = e^{im\tau}\left(-i\frac{d}{d\tau} + \frac{1}{2}m(k\cdot k)\right):e^{ik\cdot X(\tau)}:$$

PS3

from reordering, gives $h = \frac{1}{2}(k\cdot k)\ell^2$

This is a good vertex operator for $\ell^2 k^2 = 2$

same as tachyon mass-shell condition ($\alpha' k^2 = 1$)

$\alpha' = \ell^2/2$

Next :

- open string levels 1 & 2 vertex operators
- Vertex operator $\longleftrightarrow$ physical states