B4.2 Functional Analysis II
Lecture 12

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In the last lecture

- Convergence of Fourier series in L^2 spaces: Completeness of the trigonometric system.

In this lecture

- Divergence of (some) Fourier series in $C_{per}(\mathbb{R})$.
- Completeness of the trigonometric system in $C_{per}(\mathbb{R})$.
- Completeness of the trigonometric system in $L^p(-\pi,\pi)$ for $1 \leq p < \infty$.
- Convergence of Fourier series in $L^p(-\pi,\pi)$ for $1 < p < \infty$.

Partial Fourier sums and the Dirichlet kernels

Recall that the partial Fourier sums can be written as an integral operator:

$$S_N f(x) = \sum_{n=-N}^{N} c_n e^{inx} = \int_{-\pi}^{\pi} f(t)\, k_N(x-t)\, dt,$$

where

$$k_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

Partial Fourier sums and the Dirichlet kernels

Lemma

The sequence (k_N) is unbounded in $L^1(-\pi,\pi)$:

$$\|k_N\|_{L^1(-\pi,\pi)} \geq \frac{1}{C}\ln N \to \infty \text{ as } N \to \infty.$$

Proof

- Since $k_N(x) = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$ and $\sin\frac{x}{2} \leq \frac{x}{2}$ for $x > 0$, we have

$$\begin{aligned}
\|k_N\|_{L^1(-\pi,\pi)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\sin(N+\frac{1}{2})x|}{|\sin\frac{x}{2}|}dx \\
&\geq \frac{1}{\pi}\int_{-\pi}^{\pi}\Big|\sin(N+\frac{1}{2})x\Big|\frac{dx}{|x|} \\
&= \frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi}|\sin x|\frac{dx}{|x|}.
\end{aligned}$$

Partial Fourier sums and the Dirichlet kernels

Proof

- ... $\|k_N\|_{L^1(-\pi,\pi)} \geq \dfrac{2}{\pi}\displaystyle\int_0^{(N+\frac{1}{2})\pi} |\sin x| \frac{dx}{|x|}$.
- We then consider only places where $|\sin x| > \frac{1}{2}$, i.e. where $|x - (k+\frac{1}{2})\pi| < \frac{\pi}{6}$ where $k = 0, 1, \ldots$
  We then have
$$\|k_N\|_{L^1(-\pi,\pi)} \geq \frac{1}{\pi}\sum_{k=0}^{N}\int_{(k+\frac{1}{3})\pi}^{(k+\frac{1}{2})\pi} \frac{dx}{|x|} = \frac{1}{\pi}\sum_{k=0}^{N} \ln\frac{k+\frac{1}{2}}{k+\frac{1}{3}}.$$
- Now as $\ln\dfrac{k+\frac{1}{2}}{k+\frac{1}{3}} = \ln(1 + \dfrac{1}{6k+2}) \approx \dfrac{1}{6k+2}$ for large k, we have
$$\ln\frac{k+\frac{1}{2}}{k+\frac{1}{3}} \geq \frac{1}{C(k+1)}$$
  for all k and for some constant C (independent of k).

Partial Fourier sums and the Dirichlet kernels

Proof

- ... $\|k_N\|_{L^1(-\pi,\pi)} \geq \dfrac{1}{\pi} \displaystyle\sum_{k=0}^{N} \ln \frac{k+\frac{1}{2}}{k+\frac{1}{3}}$
- $\ln \dfrac{k+\frac{1}{2}}{k+\frac{1}{3}} \geq \dfrac{1}{C(k+1)}$.
- We thus have

$$\|k_N\|_{L^1(-\pi,\pi)} \geq \frac{1}{C} \sum_{k=0}^{N} \frac{1}{k+1}$$

The conclusion follows from the fact that

$$\sum_{k=0}^{N} \frac{1}{k+1} \approx \ln N \text{ for large } N.$$

Divergence of Fourier series in $C_{per}(\mathbb{R})$

Let $C_{per}(\mathbb{R})$ be the Banach space of 2π-periodic continuous function on $\mathbb{R}$, equipped with the supremum norm.

Theorem

For every given point $x_0 \in \mathbb{R}$, there exists $f \in C_{per}(\mathbb{R})$ such that $(S_N f(x_0))$ is divergent. In particular, S_N does not converges strongly in $\mathscr{B}(C_{per}(\mathbb{R}))$.

Remark

One should be reminded of the following results from Prelims: If f is 2π-periodic and piecewise differentiable, then

$$S_N f(x) \to \frac{f(x^+) + f(x^-)}{2}.$$

The above theorem shows that the hypothesis of piecewise differentiability cannot be simply dropped. We will see in the next lecture that it can however be substantially relaxed.

Divergence of Fourier series in $C_{per}(\mathbb{R})$

Proof

- We take $x_0 = 0$ and suppose by contradiction that $(S_N f(0))$ is convergent for all $f \in X = C_{per}(\mathbb{R})$.
- For $f \in X$, let $A_N f = S_N f(0)$ so that $A_N \in X^*$. Then $(A_N f)$ is convergent and hence bounded for all $f \in X$.
- By the principle of uniform boundedness, we thus have that (A_N) is bounded in X^*.
- Now recall that

$$A_N f = S_N f(0) = \int_{-\pi}^{\pi} f(x) k_N(x)\, dx.$$

  We'll be done once we show that $\|A_N\|_* = \|k_N\|_{L^1(-\pi,\pi)}$, as the latter is unbounded.

Divergence of Fourier series in $C_{per}(\mathbb{R})$

Proof

- ... $A_N f = S_N f(0) = \int_{-\pi}^{\pi} f(x) k_N(x)\, dx.$
- ... We'll be done once we show that $\|A_N\|_* = \|k_N\|_{L^1(-\pi,\pi)}$, as the latter is unbounded.
- It is clear that $|A_N f| \leq \|f\|_{sup} \|k_N\|_{L^1}$.
- For the reverse inequality, take some small $\varepsilon > 0$ and consider the continuous function f which takes value $f(x) = sign(k_N(x))$ if $|k_N(x)| \geq \varepsilon$ and is linear in the intervals where $|k_N(x)| \leq \varepsilon$. We then have $\|f\|_{sup} = 1$ and

$$A_N f \geq \int_{|k_N| \geq \varepsilon} |k_N(x)|\, dx - \int_{|k_N| < \varepsilon} |f||k_N| \geq \|k_N\|_{L^1} - CN\varepsilon.$$

This implies that $\|A_N\| \geq |A_N f| \geq \|k_N\|_{L^1} - CN\varepsilon$. Sending $\varepsilon \to 0$, we get the desired inequality.

Completeness of the trigonometric system in C_{per} and L^p

However, we have the following result:

Theorem

The sequence $\{e_n\}$ is a basis for $C_{per}(\mathbb{R})$, i.e. $Span(\{e_n\})$ is dense in $C_{per}(\mathbb{R})$.

In order words, every function in $C_{per}(\mathbb{R})$ is uniformly approximated by a sequence of trigonometric polynomials. (The approximants cannot always be the partial Fourier sums in view of the previous theorem.) Since $C_{per}(\mathbb{R})$ is dense in $L^p(-\pi, \pi)$, this implies immediately that

Theorem

For $1 \leq p < \infty$, the sequence $\{e_n\}$ is a basis for $L^p(-\pi, \pi)$, i.e. $Span(\{e_n\})$ is dense in $L^p(-\pi, \pi)$.

Completeness of the trigonometric system in C_{per}

- This is an application of the Stone-Weierstrass theorem in periodic setting. Let $X = C_{per}(\mathbb{R})$ which is an algebra, and $T = Span(\{e_n\})$ be its subalgebra of trigonometric polynomials.
- To show that T is dense in X, we only need to show that T separate points (mod. 2π).
- But this is relatively easy to check: If $p, q \in [-\pi, \pi)$ and $|p - q| \neq \pi$, we separate p and q by $\sin(x - p)$ which is a linear combination of e^{ix} and e^{-ix}. if $|q - p| = \pi$, we separate p and q by $\cos(x - p) - 1$, which is a linear combination of 1 and $e^{\pm ix}$.

Convergence of Fourier series in L^p, $1 < p < \infty$

Theorem (Carleson-Hunt)

Let $1 < p < \infty$. For every $f \in L^p(-\pi, \pi)$, $S_N f \to f$ in $L^p(-\pi, \pi)$. In order words, (S_N) converges strongly to the identity operator.

Discussion of proof

- We knew that the set T of trigonometric polynomials is dense in $L^p(-\pi, \pi)$, and for $f \in T$, $S_N f = f$ for large N and so $S_N f \to f$.
- By the theorem about strong convergence of operators (Lecture 5), it therefore suffices to show that (S_N), considered as bounded linear operators on $L^p(-\pi, \pi)$, is a bounded sequence.

Convergence of Fourier series in L^p, $1 < p < \infty$

Discussion of proof

- Recall that
$$S_N f = \int_{-\pi}^{\pi} f(t)\, k_N(x-t)\, dt \text{ where } k_N(x) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$
- For those of you who know the convolution product, the above means that $S_N f$ gives the convolution of f with the Dirichlet kernel k_N. In particular, by the so-called Young inequality
$$\|S_N f\|_{L^p} \leq \|k_N\|_{L^1} \|f\|_{L^p}.$$
Unfortunately, this is insufficient for our cause, as $\|k_N\|_{L^1}$ diverges like $\ln N$ for large N.

Convergence of Fourier series in L^p, $1 < p < \infty$

Discussion of proof

- The estimate for $\|S_N\|_{\mathscr{B}(L^p)}$ for $1 < p < 2$ is much more sophisticated. It 'interpolates' the statement that S_N is bounded in $\mathscr{B}(L^2)$ and the following estimate:

$$\int_{-\pi}^{\pi} |S_N f(x)|\, dx \leq C + C \int_{-\pi}^{\pi} |f(x)| (\max(1, \log |f(x)|))^2\, dx.$$

- The case $p > 2$ can be deduced from the case $p < 2$ by duality, which we will prove as an independent lemma below.

A duality lemma

Lemma

Suppose $1 < p < \infty$ *and* p' *is the Hölder conjugate of* p. *Then*

$$\|S_N\|_{\mathscr{B}(L^p(-\pi,\pi))} = \|S_N\|_{\mathscr{B}(L^{p'}(-\pi,\pi))}.$$

Proof

- Let $X = L^p(-\pi,\pi)$ so that $X^* = L^{p'}(-\pi,\pi)$. For $f \in X$ and $g \in X^*$, we have

$$(S_N f)g = \int_{-\pi}^{\pi} S_N f(x)\overline{g(x)}\,dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} k_N(x-t)f(t)\,dt\overline{g(x)}\,dx$$
$$= \int_{-\pi}^{\pi} f(t)\overline{\int_{-\pi}^{\pi} \overline{k_N(x-t)}g(x)\,dx}\,dt.$$

- Since k_N is real and even, we have $\overline{k_N(x-t)} = k_N(t-x)$ and so the above shows that the dual of $S_N : X \to X$ is exactly $S_N : X^* \to X^*$. The conclusion follows.

The L^1 and L^∞ case

- We now the Fourier series does not necessarily converges uniformly for continuous f. This implies that Fourier series need not converge in $L^\infty(-\pi, \pi)$ too.
- For L^1, we have the following result:

Theorem (Komolgorov)

There exists a function $f \in L^1(-\pi, \pi)$ such that $(S_N f)$ diverges everywhere. In fact, there is a set E of positive measure and a sequence N_k such that $|S_{N_k} f(x)| \to \infty$ for $x \in E$. In particular, $(S_N f)$ does not converges in L^1.

Examples

Example

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and 2π-periodic. Show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(nt) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

provided $\frac{t}{2\pi}$ is irrational.

- Let $X = C_{per}(\mathbb{R})$. For a fixed t such that $\frac{t}{2\pi} \notin \mathbb{Q}$, consider $A_N \in X^*$ be defined by $A_N f = \frac{1}{N}\sum_{n=1}^{N} f(nt)$.
- Note that $\|A_N\|_* = 1$ (check this!). Thus it suffices to show the conclusion for a f belonging to a dense subset of $C_{per}(\mathbb{R})$. By the completeness of the trigonometric system in $C_{per}(\mathbb{R})$, we only need to check for $f = e^{ikx}$, $k \in \mathbb{Z}$, in which case $A_N f$ can be explicitly summed.

Examples

Example (Riemann-Lebesgue's lemma)

For $f \in L^1(-\pi, \pi)$, there holds

$$\lim_{|n| \to \infty} \int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx = 0,$$

i.e. the Fourier coefficients (c_n) of f satisfies $c_n \to 0$ as $|n| \to \infty$.

Proof

- We saw this once in Lecture 5. Here we give a slightly different argument.
- If $f \in L^2(-\pi, \pi)$, the conclusion holds due to Bessel's inequality (Parserval's identity). We can then proceed as in the previous example, using the fact that the functional $A_n f := \int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx$ has norm $\|A_n\|_* \le 1$.

Examples

Example

Let $c_0(\mathbb{Z})$ be the space of sequence $(c_n)_{n=-\infty}^{\infty}$ such that $c_n \to 0$ as $|n| \to \infty$. Show that the map $T : L^1(-\pi, \pi) \to c_0(\mathbb{Z})$ that associates a function f to its Fourier coefficients (c_n) is not surjective.

- By the remark following the completeness of the trigonometric series in L^2, the only L^1 function whose Fourier coefficients are all zero is the zero function. Thus T is injective.
- If T was surjective, then it would be bijective and thus would have a bounded inverse by the inverse mapping theorem.
- Now let k_N be the Dirichlet kernel. We have $\|Tk_N\|_\infty = 1$ and so we would have

$$\|k_N\|_{L^1} = \|T^{-1}Tk_N\| \leq \|T^{-1}\|\|Tk_N\|_\infty = \|T^{-1}\|,$$

which is impossible as $\|k_N\|_{L^1} \sim \ln N$.