B4.2 Functional Analysis II
Lecture 11

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In the last 3 lectures

- Weak convergence.

In this lecture

- Fourier series: recap and outlook.
- Term-by-term integration and differentiation of Fourier series.
- Convergence of Fourier series in L^2 spaces: Completeness of the trigonometric system.

Fourier series

- Recall that if $f : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic, its Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \text{ and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

- We can conveniently rewrite the above using complex notation:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} =: \mathscr{F}[f]$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.$$

Fourier series

Our goal: Use FA to study the convergence/divergence of the Fourier series $\mathscr{F}[f]$ in various functional spaces.

- Strong convergence in L^p of $\mathscr{F}[f]$ to f for $f \in L^p(-\pi, \pi)$ for $1 < p < \infty$.
- Example of pointwise divergence of $\mathscr{F}[f]$ to f for continuous f.
- Pointwise convergence $\mathscr{F}[f]$ to f for Hölder continuous f.

L^2 setting

- If we write $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, then $\{e_n\}$ is an orthonormal sequence in $L^2(-\pi,\pi)$ and

$$\mathscr{F}[f] = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n.$$

- It follows that $\mathscr{F}[f] \in \overline{Span(\{e_n\})}$ for every $f \in L^2(-\pi,\pi)$.

Theorem (Completeness of the trigonometric system)

The orthonormal sequence $\{e_n\}$ is a basis for $L^2(-\pi,\pi)$, i.e. $L^2(-\pi,\pi) = \overline{Span(\{e_n\})}$. Equivalently, for every $f \in L^2(\pi,\pi)$, the Fourier series of f converges strongly in $L^2(-\pi,\pi)$ to f:

$$S_N f := \sum_{n=-N}^{N} c_n e^{inx} \xrightarrow{N\to\infty} f \text{ in } L^2.$$

Indefinite integrals

Let us do some preparation for the proof of the theorem.

Definition

The indefinite integral of a function $f \in L^1_{loc}(\mathbb{R})$ is

$$F(x) = \int_a^x f(t)\,dt \text{ for some } a \in \mathbb{R}.$$

Properties:

- F is continuous (by the dominated convergence theorem).
- F is almost everywhere differentiable: For almost all x,

  $$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

  This is a consequence of the Lebesgue differentiation theorem.
- If F is periodic then so is f. The converse needn't hold.

Indefinite integrals

Properties:

- F satisfies the integration by parts rule (Sheet 4): For smooth φ one has

$$\int_b^c F(x)\varphi'(x)\,dx = [F\varphi]_b^c - \int_b^c f(x)\varphi(x)\,dx.$$

Term-by-term differentiation of Fourier series

Theorem (Termwise differentiation of Fourier series)

Suppose that $f \in L^1_{loc}(\mathbb{R})$ and let F be the indefinite integral of f. If F is 2π-periodic and $F \sim \sum c_n e^{inx}$, then f is 2π-periodic and $f \sim \sum in\, c_n\, e^{inx}$.

Proof

- We have that $F' = f$ almost everywhere. Since F is 2π-periodic, and since
$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h},$$
$f = F'$ is also 2π-periodic.
- For the Fourier coefficients of f, we integrate by parts:
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{in}{2\pi} \int_{-\pi}^{\pi} F(x)\, e^{-inx}\, dx = in\, c_n.$$

Term-by-term integration of Fourier series

Theorem (Termwise integration of Fourier series)

Suppose that $f \in L^1(-\pi,\pi)$ is 2π-periodic and let F be the indefinite integral of f. If $f \sim \sum c_n e^{inx}$, then $F(x) - c_0\, x$ is 2π-periodic and $F(x) - c_0\, x \sim C_0 + \sum_{n\neq 0} \frac{c_n}{in}\, e^{inx}$ where C_0 is a suitable constant.

Proof

- Let $G(x) = F(x) - c_0\, x$. We have

$$G(x+2\pi) - G(x) = \int_x^{x+2\pi} f(t)\,dt - 2\pi c_0 = 2\pi c_0 - 2\pi c_0 = 0,$$

  and so G is 2π-periodic.
- Now if $G \sim \sum d_n e^{inx}$, then $f \sim \sum i n d_n e^{inx} = \sum c_n e^{inx}$ using term-by-term differentiation. Therefore $d_n = \frac{1}{in} c_n$ for $n \neq 0$.

Completeness of the trigonometric system in L^2

Theorem (Completeness of the trigonometric system)

The orthonormal sequence $\{e_n\}$ is a basis for $L^2(-\pi,\pi)$, i.e. $L^2(-\pi,\pi) = \overline{Span(\{e_n\})}$. Equivalently, for every $f \in L^2(\pi,\pi)$, the partial Fourier sums $S_N f$ of f converges strongly in $L^2(-\pi,\pi)$ to f.

Proof

- We have that $L^2(-\pi,\pi) = \overline{Span(\{e_n\})} \oplus Span(\{e_n\})^\perp$. Therefore, to conclude, we need to show that $Span(\{e_n\})^\perp = 0$, i.e. if $f \in L^2(-\pi,\pi)$ is orthogonal to all e^{inx}, then $f = 0$. Equivalently, we need to show that if all the Fourier coefficients of a function $f \in L^2(-\pi,\pi)$ are all zero, then $f = 0$ a.e.

Completeness of the trigonometric system in L^2

Proof

- ... we need to show that if all the Fourier coefficients of a function $f \in L^2(-\pi, \pi)$ are all zero, then $f = 0$ a.e.
- Let F be an indefinite integral of f. Since $\langle f, 1 \rangle = 0$,

$$F(x + 2\pi) - F(x) = \int_x^{x+2\pi} f(t)\, dt = 0.$$

  Hence F is periodic and all Fourier coefficients of F except possibly the zero-th coefficient are zero (by term-by-term integration).
  Therefore, by considering F instead of f, we may assume that f is continuous.

Completeness of the trigonometric system in L^2

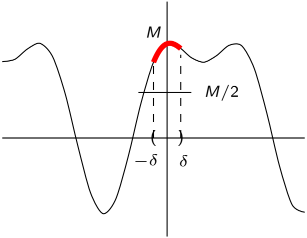
Proof

- ... we need to show that if all the Fourier coefficients of a continuous function f are all zero, then $f = 0$.
- It is enough to consider the case f is real-valued.
- Suppose by contradiction that $f \neq 0$.
  Since $|f|$ is periodic and continuous, it attains it maximum value $M > 0$ at some point, say x_0.
  Replacing f by $-f$ if necessary, we may assume that $f(x_0) = M > 0$.
  Using a translation if necessary, we may further assume that $x_0 = 0$.

Completeness of the trigonometric system in L^2

Proof

- ... we need to show that if all the Fourier coefficients of a continuous function f are all zero, then $f = 0$.
- ... $M = f(0) = \max|f|$.



- The trick is to note that $\int fP = 0$ for all trigonometric polynomials P. If we can cook up trigomometric polynomials P_n so that the integral $\int fP_n$ concentrates in an interval around 0 we'd win.

Completeness of the trigonometric system in L^2

Proof

- ... we need to show that if all the Fourier coefficients of a continuous function f are all zero, then $f = 0$.
- ... $M = f(0) = \max |f|$.
- Select $\delta > 0$ such that $f(x) > \frac{1}{2}M$ in $(-\delta, \delta) \subset (-\pi, \pi)$.
- Consider the function

$$g(x) = 1 + \cos x - \cos \delta$$

  which satisfies:
  - ▷ $g > 1$ in $(-\delta, \delta)$,
  - ▷ $|g| \leq 1$ in $(-\pi, \pi) \setminus (-\delta, \delta)$.
- These imply that $P_n = g^n$ satisfies
  - ▷ $P_n > 1$ in $(-\delta, \delta)$,
  - ▷ $\inf_{(-\delta/2, \delta/2)} P_n \to \infty$ as $n \to \infty$,
  - ▷ $|P_n| \leq 1$ in $(-\pi, \pi) \setminus (-\delta, \delta)$.

Completeness of the trigonometric system in L^2

Proof

- ... we need to show that if all the Fourier coefficients of a <u>continuous</u> function f are all zero, then $f = 0$.
- ... $f(x) > \frac{1}{2}M$ in $(-\delta, \delta) \subset (-\pi, \pi)$.
- We are now ready to deduce a contradiction:

$$
\begin{aligned}
0 &= \int_{-\pi}^{\pi} f(x)\, P_n(x)\, dx \\
&\geq \int_{-\delta/2}^{\delta/2} f(x)\, P_n(x)\, dx - \int_{(-\pi,\pi)\setminus(-\delta,\delta)} |f(x)||P_n(x)|\, dx \\
&\geq \frac{1}{2} M \inf_{(-\delta/2,\delta/2)} P_n\, \delta - 2\pi M \stackrel{n\to\infty}{\longrightarrow} \infty.
\end{aligned}
$$

We are done.

L^p setting: a remark

Remark

In the proof above, we only use the space L^2 for the decomposition $L^2(-\pi,\pi) = \overline{Span(\{e_n\})} \oplus Span(\{e_n\})^\perp$. The later part of the proof only uses integrability. We thus have: If all the Fourier coefficients of a function $f \in L^1(-\pi,\pi)$ are zero, then $f = 0$ a.e.

L^p setting: preparatory work

- It will be convenient to consider Fourier series from operator theory. Suppose we are in a (Banach) space X, which will be one of the L^p spaces, or some space of continuous functions.
- Our question can be rephrased as follows: Is it true that, for every $f \in X$,

$$S_N f := \underbrace{\sum_{n=-N}^{N} c_n e^{inx}}_{\in Span(\{e_n\})} \xrightarrow{N\to\infty} f \text{ where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx?$$

- Equivalently, is it true that (S_N), considered as a sequence of bounded linear operators on X, converges strongly (i.e. pointwise) to the identity operator on X?
- Note that, unlike the L^2 case, there is a subtle difference between the above question and the related question if $\{e_n\}$ is a basis for X, i.e. if $Span(\{e_n\})$ is dense in X.

Partial Fourier sums and the Dirichlet kernels

- The operator S_N can be written as an integral operator:

$$S_N f = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \right\} e^{inx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-N}^{N} e^{in(x-t)}\, dt$$

$$= \int_{-\pi}^{\pi} f(t)\, k_N(x-t)\, dt,$$

where

$$k_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

The functions k_N's are called the Dirichlet kernels.