B4.2 Functional Analysis II
Lecture 15

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In the last lecture

- The ABCs of spectral theory for bounded linear operators.
- Spectra of normal operators.

In this lecture

- Spectra of self-adjoint operators.
- Spectra of unitary operators.

Self-adjoint operators

Theorem

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be self-adjoint. Then

1. $\sigma_r(T) = \emptyset$ *and* $\sigma(T) = \sigma_{ap}(T) = \sigma_p(T) \cup \sigma_c(T)$,
2. *If x and y are eigenvectors of T corresponding to different eigenvalues, then* $\langle x, y \rangle = 0$.
3. $rad(\sigma(T)) = \|T\|$,
4. $\sigma(T) \subset [a, b] \subset \mathbb{R}$ *where*

$$a = \inf_{\|x\|=1} \langle Tx, x \rangle \text{ and } b = \sup_{\|x\|=1} \langle Tx, x \rangle.$$

Furthermore, the endpoints a and b belong to the spectrum of T.

Spectral radius of self-adjoint operators

Points (i) and (ii) are consequence of the fact that T is normal. We will discuss (iii) and (iv) separately.

Lemma

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be self-adjoint. Then

$$rad(\sigma(T)) = \|T\|.$$

Proof

- By Gelfand's formula

$$\mathrm{rad}(\sigma(T)) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

- As T is self-adjoint, $\|T^2\| = \|T\|^2$ (Lecture 4). A simple induction thus gives $\|T^n\| = \|T\|^n$ when $n = 2^k$, $k \in \mathbb{N}$. The conclusion follows.

Spectral radius of normal operators

It turns out that the result holds for normal operators with a little bit more work (Sheet 4).

Lemma

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be normal. Then

$$rad(\sigma(T)) = \|T\|.$$

Idea: Use $\|(T^*)^n T^n\| = \|T^n\|^2$ (Lecture 4) and revisit the proof of the self-adjoint case with the self-adjoint operator $T^* T$.

Spectra of self-adjoint operators are real

Lemma

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be self-adjoint. Then $\sigma(T) \subset \mathbb{R}$.

Proof

- Since T is self-adjoint, we have

$$\langle Tx, x\rangle = \langle x, T^* x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \text{ for all } x.$$

  This means that $\langle Tx, x\rangle$ is real for all $x \in X$.
  The result thus follows from the statement obtained in an example in Lecture 14: $\sigma(T)$ is a subset of the closure of $\{\langle Tx, x\rangle : \|x\| = 1\}$.

Spectra of self-adjoint operators are real

Proof

- Let us have a more direct argument. We know that $\sigma(T) = \sigma_{ap}(T)$.
- Consider $\sigma_p(T)$: If λ is an eigenvalue of T with a unit eigenvector x, then $\lambda = \langle \lambda x, x\rangle = \langle Tx, x\rangle \in \mathbb{R}$.
  (Note that, though we knew $\sigma_p(T) = \sigma_p'(T^*) = \sigma_p'(T)$, this is insufficient to say that $\sigma_p(T) \subset \mathbb{R}$.)
- The case of approximate eigenvalue requires only a little bit more effort:
  - ⋆ Suppose $\lambda \in \sigma_{ap}(T) = \sigma(T)$ and select $(x_n) \subset X$ such that $\|x_n\| = 1$ and $\lambda x_n - Tx_n \to 0$.
  - ⋆ By Cauchy-Schwarz' inequality, $\langle \lambda x_n - Tx_n, x_n\rangle \to 0$.
  - ⋆ Hence $\langle Tx_n, x_n\rangle \to \lambda$.
    Since the left hand side is real, we must have that λ is real too.

On the location of spectra of self-adjoint operators

Theorem

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$. If T is self-adjoint, then the spectrum of T lies in the closed interval $[a, b]$ on the real axis, where

$$a = \inf_{\|x\|=1} \langle Tx, x\rangle \text{ and } b = \sup_{\|x\|=1} \langle Tx, x\rangle.$$

Furthermore, the endpoints a and b belong to the spectrum of T.

Proof

- We knew that for every $\lambda \in \sigma(T)$ there exists $(x_n) \subset X$, $\|x_n\| = 1$ such that $\langle Tx_n, x_n\rangle \to \lambda$. This implies that $\sigma(T) \subset [a, b]$.

On the location of spectra of self-adjoint operators

Proof

- It remains to show that $a, b \in \sigma(T)$.
- By considering $-T$ in place of T, we may assume without loss of generality that $|a| \leq |b|$.
- We know that $\mathrm{rad}(\sigma(T)) = \|T\|$, and so by the definition of b, $\mathrm{rad}(\sigma(T)) = \|T\| \geq |b| \geq |a|$.
  Since $\sigma(T)$ is closed and is contained in $[a, b]$, this implies that $\|T\| = |b|$ and
  - ⋆ If $|a| < |b|$, then b belongs to $\sigma(T)$,
  - ⋆ If $|a| = |b|$, then at least a or b belongs to $\sigma(T)$.
- Now applying the statement we just proved to $\tilde{T} = cI + T$
  - ⋆ with $c \gg 1$ such that $c + b > c + a > 0$, we get $c + b \in \sigma(\tilde{T})$ and so $b \in \sigma(T)$,
  - ⋆ and with $c \ll -1$ such that $c + a < c + b < 0$, we get $c + a \in \sigma(\tilde{T})$ and so $a \in \sigma(T)$.

Unitary operators

Proposition

Let X be a complex Hilbert space and $U \in \mathscr{B}(X)$ be unitary. Then $\sigma(U) \subset \{|\lambda| = 1\}$.

Proof

- Since U is unitary, $\|U\| = 1$. It follows that $\mathrm{rad}(\sigma(U)) \leq \|U\| = 1$, i.e. $\sigma(U) \subset \{|\lambda| \leq 1\}$.
- Applying the above to $U^* = U^{-1}$, we have $\sigma(U^{-1}) \subset \{|\lambda| \leq 1\}$.
- As $\sigma(U^{-1}) = \sigma(U)^{-1}$, we deduce that $\sigma(U) \subset \{|\lambda| \leq 1\} \cap \{|\lambda| \geq 1\} = \{|\lambda| = 1\}$.

Example 1

Example

Let $X = \ell^2$ and $T((a_1, a_2, a_3, \ldots)) = (m_1 a_1, m_2 a_2, m_3 a_3, \ldots)$ where $(m_1, m_2, m_3, \ldots)$ is a given bounded sequence. Compute the different spectra of T.

- Note that $T^*((a_1, a_2, a_3, \ldots)) = (\bar{m}_1 a_1, \bar{m}_2 a_2, \bar{m}_3 a_3, \ldots)$. It follows that T^* and T commute, i.e. T is normal. Hence $\sigma_r(T) = \emptyset$ and $\sigma(T) = \sigma_{ap}(T) = \sigma_p(T) \cup \sigma_c(T)$.
- It is easy to see that if $\{e_1, e_2, \ldots\}$ is the standard basis of ℓ^2, then $Te_k = m_k e_k$. Hence $m_k \in \sigma_p(T)$.
- Conversely, if λ is an eigenvalue of T with eigenvector $(a_1, a_2, \ldots)$, then
$$(\lambda - m_k)a_k = 0 \text{ for all } k.$$
Since there is at least one non-zero a_k's, we have that $\lambda = m_k$ for some k. So $\sigma_p(T) = \{m_1, m_2, \ldots\} =: S$.

Example 1

- Since $\sigma(T)$ is closed, we have $\sigma(T) \supset \bar{S} = \overline{\{m_1, m_2, \ldots\}}$.
- We claim that $\sigma(T) = \bar{S}$. Indeed, if $dist(\lambda, S) > c > 0$, then $\lambda I - T$ is invertible

$$(\lambda I - T)^{-1}(b_1, b_2, \ldots) = \Big(\frac{1}{\lambda - m_1} b_1, \frac{1}{\lambda - m_2} b_2, \ldots\Big) \in \ell^2$$

since

$$\sum_{k=0}^{\infty} \frac{1}{|\lambda - m_k|^2} |b_k|^2 \leq \frac{1}{c^2} \sum_{k=0}^{\infty} |b_k|^2 < \infty.$$

- We conclude that $\sigma(T) = \sigma_{ap}(T) = \bar{S}$, $\sigma_p(T) = S$, $\sigma_c(T) = \bar{S} \setminus S$ and $\sigma_r(T) = \emptyset$.

Example 2

Example

Let $X = L^2(\mathbb{R})$ and consider the multiplication operator M_h where $h \in L^\infty(\mathbb{R})$, i.e. $M_h f = hf$. Compute the different spectra of M_h.

- This is similar to the previous example, but more involved.
- Again, we have that $M_h^* = M_{\bar{h}}$ and so M_h is normal. Hence $\sigma_r(T) = \emptyset$ and $\sigma(T) = \sigma_{ap}(T) = \sigma_p(T) \cup \sigma_c(T)$.
- In the same fashion one can show that

$$\sigma_p(M_h) = \{\lambda \in \mathbb{R} : \{h = \lambda\} \text{ has positive measure}\}$$

where for each $\lambda \in \sigma_p(M_h)$, the corresponding eigenfunctions are those supported on the set $\{h = \lambda\}$.

Example 2

- A difference in the treatment occur in the identification of the discussion of the full spectrum. It turns out that

$$\begin{aligned}\sigma(M_h) = \mathrm{Ess}\ \mathrm{Im}(h) := &\text{ the essential range of } h\\ &= \Big\{\lambda \in \mathbb{R} : h^{-1}((\lambda - \epsilon, \lambda + \epsilon)) \text{ has positive}\\ &\qquad \text{measure for all small } \epsilon > 0\Big\}\end{aligned}$$

- The proof that if $\lambda \notin \mathrm{Ess}\ \mathrm{Im}(h)$ then $\lambda I - M_h$ is invertible remains similar.
  However $\mathrm{Ess}\ \mathrm{Im}(h)$ needs not be the same as the closure of $\sigma_p(T)$, so we need to conclude differently.

Example 2

- We pick $\lambda \in \mathrm{Ess\ Im}(h)$ and show that λ is an approximate eigenvalue.
  - ⋆ Select a measurable subset Z_n of $h^{-1}((\lambda - 1/n, \lambda + 1/n))$ with positive measure and let
  $$f_n = \frac{1}{|Z_n|^{1/2}} \chi_{Z_n}$$
  so that $\|f_n\|_{L^2} = 1$.
  - ⋆ Then $|(\lambda I - M_h) f_n| \leq \frac{1}{n} |f_n| \to 0$ in X.
- We conclude that

$$\begin{aligned}
\sigma(M_h) &= \sigma_{ap}(M_h) = \mathrm{Ess\ Im}(h), \\
\sigma_p(M_h) &= \{\lambda \in \mathbb{R} : \{h = \lambda\} \text{ has positive measure}\}, \\
\sigma_r(M_h) &= \emptyset, \\
\sigma_c(M_h) &= \sigma_{ap}(M_h) \setminus \sigma_p(M_h).
\end{aligned}$$

Example 3

Example

Let T denote the shift operator $Tf(x) = f(x+1)$ for $f \in X = L^2(\mathbb{R})$. Compute the different spectra of T.

- T is a unitary operator. So $\sigma_r(T) = \emptyset$ and $\sigma(T) = \sigma_{ap}(T) \subset \{|\lambda| = 1\}$.
- Suppose $|\lambda| = 1$ and $Tf = \lambda f$. Then $f(x+n) = \lambda^n f(x)$ and

$$\int_{\mathbb{R}} |f|^2\,dx = \sum_{n=-\infty}^{\infty} \int_0^1 |f(x+n)|^2\,dx = \sum_{n=-\infty}^{\infty} \int_0^1 |f(x)|^2\,dx.$$

Since $f \in L^2$, this is possibly only if $f = 0$. Therefore $\sigma_p(T) = \emptyset$ and $\sigma(T) = \sigma_c(T)$.

Example 3

- Finally, we show that if $|\lambda| = 1$, then $\lambda \in \sigma_{ap}(T)$.
  - ⋆ For $n = 1, 2, \ldots$, we take

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{2n}}\lambda^k & \text{if } x \in [k, k+1), -n \leq k \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

  - ⋆ Then $f_n \in X$, $\|f_n\| = 1$,

$$Tf_n = \lambda f_n \text{ in } \mathbb{R} \setminus ([-n-1, -n] \cup [n-1, n]),$$

and

$$\|\lambda f_n - Tf_n\|^2 = \frac{1}{n} \to 0.$$

- We conclude that $\sigma(T) = \sigma_{ap}(T) = \sigma_c(T) = \{|\lambda| = 1\}$ and $\sigma_p(T) = \sigma_r(T) = \emptyset$.