B4.2 Functional Analysis II
Lecture 10

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In the last 2 lectures

- Weak convergence: Examples and basic properties.
- Mazur's theorem.

In this lecture

- Weak sequential compactness property.

Recap

Suppose that X is a normed vector space and $(x_n) \subset X$ converges weakly to x. We knew:

- (x_n) is bounded.
- $\|x\| \leq \liminf \|x_n\|$.
- a finite convex linear combination of x_n's converges strongly to x.

Weak sequential compactness property

Definition

A subset A of a normed vector space X is called weakly sequentially compact if every sequence of A has a subsequence weakly convergent to a point of A.

Theorem (Weak sequential compactness in reflexive Banach spaces)

The closed unit ball of a reflexive Banach space is weakly sequentially compact. In particular, every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

The converse is also true and is known as Eberlein's theorem: The closed unit ball in a Banach space X is weakly sequentially compact *only if* X is reflexive.

Weak sequential compactness property

Proof when X is a Hilbert space.

- We will only prove the theorem in the case of Hilbert spaces.
- Suppose (x_n) is a sequence in the unit ball. We want to extract a weakly convergent subsequence (x_{n_j}). We will do a bit of reverse engineering before going down with the construction.
- By the Riesz representation theorem, the perspective sequence converges weakly if and only if there exists an x_* such that

$$\langle x_{n_j}, x\rangle \to \langle x_*, x\rangle \text{ for all } x \in X.$$

- It suffices to have $(\langle x_{n_j}, x\rangle)$ converges for all $x \in X$, since if this is the case, the function $x \mapsto \lim \overline{\langle x_{n_j}, x\rangle}$ is bounded linear and so can be expressed as $\overline{\langle x_*, x\rangle}$.

Weak sequential compactness property

- ...it suffices to have

$$(\langle x_{n_j}, x\rangle) \text{ converges for all } x \in X.$$

- There is an obvious set of x's for which $(\langle x_{n_j}, x\rangle)$ converges: the x's which are orthogonal to the x_n's.
- Let Y be the span of x_n's and $\bar{Y}$ be its closure (i.e. the closed linear span of the x_n's). Then, for every $z \in \bar{Y}^\perp$, $(\langle x_n, z\rangle)$ is the constant sequence (0) and hence convergent.
- By the projection theorem, $X = \bar{Y} \oplus \bar{Y}^\perp$. Thus, if we can arrange a subsequence (x_{n_j}) so that $(\langle x_{n_j}, y\rangle)$ converges for every $y \in \bar{Y}$, we will be done.
  Indeed, for $x \in X$, we can write $x = y + z \in \bar{Y} \oplus \bar{Y}^\perp$ and so

$$(\langle x_{n_j}, x\rangle) = (\langle x_{n_j}, y\rangle) \text{ converges.}$$

Weak sequential compactness property

- ...it suffices to have

$$(\langle x_{n_j}, y\rangle) \text{ converges for all } y \in \bar{Y}.$$

- Since $\bar{Y}$ is the closure of Y, it suffices (check this!) to have

$$(\langle x_{n_j}, y\rangle) \text{ converges for all } y \in Y.$$

  This means

$$(\langle x_{n_j}, x_m\rangle) \text{ converges for all } m.$$

- We use a diagonal process.
  - ⋆ To begin with, we note that the sequence $(\langle x_n, x_1\rangle)$ is bounded: $|\langle x_n, x_1\rangle| \leq \|x_n\|\|x_1\| \leq 1$.
    By the Bolzano-Weierstrass lemma, we can extract a subsequence $n_j^{(1)}$ such that $(\langle x_{n_j^{(1)}}, x_1\rangle)$ is convergent.

Weak sequential compactness property

- ... it suffices to have $(\langle x_{n_j}, x_m \rangle)$ converges for all m.
  - ... we can extract a subsequence $n_j^{(1)}$ such that $(\langle x_{n_j^{(1)}}, x_1 \rangle)$ is convergent.
  - We then consider $(\langle x_{n_j^{(1)}}, x_2 \rangle)$ and select a convergent subsequence $(\langle x_{n_j^{(2)}}, x_2 \rangle)$.
    Clearly, $\langle x_{n_j^{(2)}}, x_1 \rangle$ is also convergent.
  - Proceeding in this way, we constructed nested subsequence $(n_j^{(k)})$ such that $\langle x_{n_j^{(k)}}, x_m \rangle$ is convergent (with respect to j) for every $m \leq k$.
  - Finally, let $x_{n_j} = x_{n_j^{(j)}}$. . For every fixed m, $(n_j)_{j \geq m}$ is then a subsequence of $(n_j^{(m)})_{j \geq m}$.
    Therefore $(\langle x_{n_j}, x_m \rangle)$ is convergent for every m as desired.

Application 1: Closest point in a closed convex subset

Theorem (Closest point in a closed convex subset)

Let K be a non-empty closed convex subset of a reflexive Banach space X. Then, for every $x \in X$, there is a point $y \in K$ such that no other point in K which is closer to x than y.

Proof

- Let $x_n \in K$ be such that $\|x - x_n\| \to d := dist(x, K)$.
- When X is Hilbert, the proof went by using the parallelogram law to show that (x_n) is Cauchy and hence strongly converges to a limit $x_* \in K$ which satisfies $\|x - x_*\| = d$.

Application 1: Closest point in a closed convex subset

Proof

- Let $x_n \in K$ be such that $\|x - x_n\| \to d := dist(x, K)$.
- In the present case, we argue as follow: The sequence (x_n) is bounded and the space X is reflexive. Hence, by the weak sequential compactness property, (x_n) has a weakly convergent subsequence (x_{n_j}).
- Let x_* be the weak limit of (x_{n_j}). By Mazur's theorem, $x_* \in K$. In particular, $\|x - x_*\| \geq d$.
  On the other hand, by the weak lower semi-continuity of norm, we have
  $$\|x - x_*\| \leq \liminf_{j \to \infty} \|x - x_{n_j}\| = d.$$
  We conclude that $\|x - x_*\| = d$, i.e. x_* is the desired point y.

Application 1: Closest point in a closed convex subset

Remark

When X is uniformly convex, the closest point y is unique (Check this!). This does not necessarily hold in general.

- Let X be $\mathbb{R}^2$ equipped with the norm

$$\|(x_1, x_2)\|_\infty = \max(|x_1|, |x_2|).$$

- Let K be the closed unit ball in X, i.e. K is the square $[-1, 1] \times [-1, 1]$, and $x = (3, 0)$.
- The distance $dist(x, K)$ is readily found to be 2, and for every point y on the right side of K, i.e. $y = (1, y_2)$ with $|y_2| \leq 1$, we have $\|x - y\| = 2 = dist(x, K)$.

Application 2: Convergence a.e. and of norms $\Rightarrow$ Strong convergence in L^p

Lemma

Let $E \subset \mathbb{R}^n$ be measurable and $1 \leq p < \infty$. Suppose that $(f_k) \subset L^p(E)$ is a bounded sequence such that $\|f_k\|_{L^p} \to \|f\|_{L^p}$ and $f_k \to f$ a.e. in E. Then $f_k \to f$ in $L^p(E)$.

Proof

- When $p = 1$, the conclusion follows from the limit

$$\int_E (|f_k| - |f_k - f|) \to \int_E |f|,$$

which is true by the dominated convergence theorem and the triangle inequality $||f_k| - |f_k - f|| \leq |f|$, and the fact that f is integrable (due to Fatou's lemma).

Application 2: Convergence a.e. and of norms $\Rightarrow$ Strong convergence in L^p

Proof

- Consider now the case $p \in (1, \infty)$.
  - ⋆ $L^p(E)$ is reflexive: the weak sequential compactness property holds.
- Suppose $f_k \not\rightharpoonup f$ in $L^p(E)$. Then $\exists\, g \in L^{p'}(E) \cong (L^p(E))^*$ such that
$$\int_E f_k g \not\to \int_E fg.$$
It follows that $\exists\, \varepsilon > 0$ and a subsequence (f_{k_j}) such that
$$\Big|\int_E (f_{k_j} - f)g\Big| > \varepsilon.$$

Application 2: Convergence a.e. and of norms $\Rightarrow$ Strong convergence in L^p

Proof

- Suppose not ... $\exists$ (f_{k_j}) such that
$$\Big|\int_E (f_{k_j} - f)g\Big| > \varepsilon.$$
- Since (f_{k_j}) is bounded in $L^p(E)$, the weak sequential compactness property gives another subsequence $(f_{k_j'})$ which converges weakly in $L^p(E)$.
- We knew weak limits coincide with a.e. limits if both exist. So $f_{k_j'} \rightharpoonup f$ in $L^p(E)$ and
$$\int_E (f_{k_j'} - f)g \to 0,$$
which gives a contradiction!