B4.2 Functional Analysis II
Lecture 6

Luc Nguyen
luc.nguyen@maths

University of Oxford

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In the last lecture

- The Baire category theorem.
- The principle of uniform boundedness.

In this lecture

- The open mapping theorem.
- The inverse mapping theorem.

The open mapping theorem

Theorem (Open mapping theorem)

Let $T : X \to Y$ be a bounded linear operator from a Banach space X onto another Banach space Y. Then T is an open map, i.e. images of open sets are open.

Remark

- *It should be clear that the theorem is wrong if T wasn't surjective: If $\mathrm{Im}\, T$ is a proper subspace of Y, then it cannot contain an open ball.*
- *Note that one should not confuse the conclusion with the statement that preimages of open sets are open, which is clear because of the continuity of T.*

The open mapping theorem

Proof

- The proof will be split into several steps:
  1. Use the Baire category theorem to show that $\overline{T(B_X(0,1))}$ contains an open ball $B_Y(y_0, r_0)$.
  2. Use convexity/symmetricity to show that $\overline{T(B_X(0,1))}$ contains $B_Y(0, r_0)$.
  3. Show that $T(B_X(0,2)) \supset \overline{T(B_X(0,1))} \supset B_Y(0, r_0)$.
  4. Wrap up using linearity.
- Step 1: We show that $\overline{T(B_X(0,1))}$ contains an open ball $B_Y(y_0, r_0)$.
  - ⋆ As T is *surjective*, $Y = \cup_{n=1}^{\infty} T(B_X(0,n))$.
  - ⋆ Since *Y is complete*, we have by the Baire category theorem that some $T(B_X(0,n_0))$ is not nowhere dense, i.e. $\overline{T(B_X(0,n_0))}$ has non-empty interior.
  - ⋆ Therefore we can take an open ball $B_Y(n_0 y_0, n_0 r_0) \subset \overline{T(B_X(0,n_0))}$. Step 1 follows by linearity.

The open mapping theorem

Proof

- Step 2: We show that $\overline{T(B_X(0,1))} \supset B_Y(0, r_0)$.
  - ⋆ By Step 1, $\overline{T(B_X(0,1))} \supset B_Y(y_0, r_0)$.
  - ⋆ Note that, by linearity, if $y \in \overline{T_X(B(0,1))}$, then $-y \in \overline{T(B_X(0,1))}$.
  - ⋆ It follows that $\overline{T(B_X(0,1))} \supset B_Y(-y_0, r_0)$.
  - ⋆ Now as $\overline{T(B_X(0,1))}$ is convex (check this) and every point of $B_Y(0, r_0)$ is the midpoint of a line segment connecting a point in $B_Y(y_0, r_0)$ and a point in $B_Y(-y_0, r_0)$, we have that $\overline{T(B_X(0,1))} \supset B_Y(0, r_0)$, which concludes Step 2.
- Step 3: We show that $T(B_X(0,2)) \supset \overline{T(B_X(0,1))}$.
  - ⋆ This is perhaps the trickiest part of the proof.
  - ⋆ Take an arbitrary $y \in \overline{T(B_X(0,1))}$. We will show that

$$y = \sum Tx_k \text{ where } x_k \in B_X(0, 2^{1-k}),$$

which will imply that $y = Tx$ where $x = \sum x_k \in B_X(0,2)$.

The open mapping theorem

Proof

- Step 2: We show that $\overline{T(B_X(0,1))} \supset B_Y(0,r_0)$.
- Step 3: We show that $T(B_X(0,2)) \supset \overline{T(B_X(0,1))}$.
  - ⋆ Suppose we start by selecting $x_1 \in B_X(0,1)$ such that Tx_1 approximates y with some precision ε_1 to be tuned:

$$\|y - Tx_1\| < \varepsilon_1.$$

  - ⋆ By dilating the statement of Step 2, this gives

$$y - Tx_1 \in B_Y(0,\varepsilon_1) = \frac{\varepsilon_1}{r_0} B_Y(0,r_0) \subset \frac{\varepsilon_1}{r_0}\overline{T(B_X(0,1))} = \overline{T(B_X(0,\frac{\varepsilon_1}{r_0}))}.$$

  - ⋆ Repeating this process, we pick $x_2 \in B_X(0,\frac{\varepsilon_1}{r_0})$ such that Tx_2 approximates $y - Tx_1$ with some precision ε_2:

$$\|y - Tx_1 - Tx_2\| < \varepsilon_2 \text{ and } y - Tx_1 - Tx_2 \in \overline{T(B_X(0,\frac{\varepsilon_2}{r_0}))}.$$

The open mapping theorem

Proof

- Step 2: We show that $\overline{T(B_X(0,1))} \supset B_Y(0, r_0)$.
- Step 3: We show that $T(B_X(0,2)) \supset \overline{T(B_X(0,1))}$.
  - ⋆ So inductively, for any given sequence of precision (ε_k), we can select a sequence $(x_k) \subset X$ with $x_k \in B_X(0, \frac{\varepsilon_{k-1}}{r_0})$ and

$$\left\| y - \sum_{k=1}^{n} Tx_k \right\| < \varepsilon_n \text{ and } y - \sum_{k=1}^{n} Tx_k \in \overline{T(B_X(0, \frac{\varepsilon_n}{r_0}))}.$$

  - ⋆ So as long as $\varepsilon_k \to 0$, we have $y = \sum Tx_k$, and as long as $\sum \varepsilon_k < \infty$, the series $\sum x_k$ converges (since *X is complete*).
  - ⋆ The conclusion of Step 3 is obtained by the convenient choice $\varepsilon_k = r_0 2^{-k}$.

The open mapping theorem

Proof

- Step 2: We show that $\overline{T(B_X(0,1))} \supset B_Y(0, r_0)$.
- Step 3: We show that $T(B_X(0,2)) \supset \overline{T(B_X(0,1))}$.
- Step 4: We now wrap up the proof.
  - ⋆ Suppose U is an open set in X. We need show that $T(U)$ is open, i.e. for every $y \in T(U)$, $T(U)$ contains an open ball centered at y.
  - ⋆ Pick $x \in U$ so that $Tx = y$. As U is open, U contains some open ball $B_X(x, r)$.
  - ⋆ By linearity, we have $T(B_X(x,s)) = y + \frac{s}{2} T(B_X(0,2))$ (check this!). Hence, by Step 2 and Step 3,

$$T(U) \supset T(B_X(x,r)) \supset y + \frac{s}{2} B_Y(0, r_0)) = B_Y(y, r_0 s/2),$$

as wanted.

The inverse mapping theorem

Theorem (Inverse mapping theorem)

A bounded bijective linear operator of a Banach space onto another has a bounded inverse.

Proof

- Let X, Y be Banach spaces and $T \in \mathscr{B}(X, Y)$ be bijective. Let T^{-1} be its algebraic inverse map.
- It is clear that T^{-1} is linear.
- By the open mapping theorem, T maps open sets to open sets. This means that the pre-images under T^{-1} of open sets are open, i.e. T^{-1} is continuous. Hence $T^{-1} \in \mathscr{B}(Y, X)$.

Application 1: equivalence of norms

Example

Let X be a Banach space with respect to two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and suppose that there is a constant $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$. Then the two norms are equivalent, i.e. there is a constant C' such that $\|x\|_2 \leq C'\|x\|_1$ for all $x \in X$.

Proof: Apply the inverse mapping theorem to the identity operator.

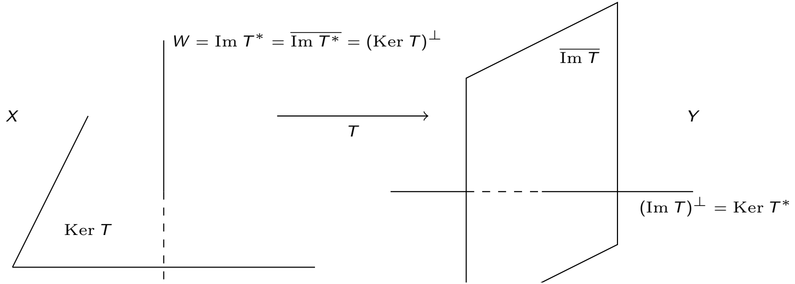
Application 2: closed operators

Theorem

Let $T \in \mathscr{B}(X, Y)$ be a bounded linear operators between Hilbert spaces. Then TX is closed if and only if T^*Y is closed.

Proof:

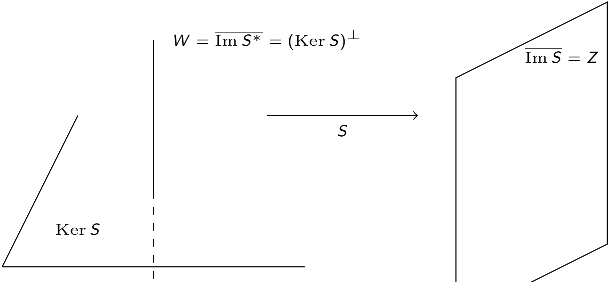
- It suffices to show to show only one direction, as $T^{**} = T$.
- ($\Leftarrow$) Suppose that $W = \mathrm{Im}\ T^*$ is closed in X. We need to show that TX is closed in X.



Application 2: closed operators

Proof:

- To get rid of the cokernel, we restrict attention to the closure of the range $Z = \overline{\mathrm{Im}\, T}$: i.e. let $S \in \mathscr{B}(X, Z)$ be given by $Sx = Tx$.



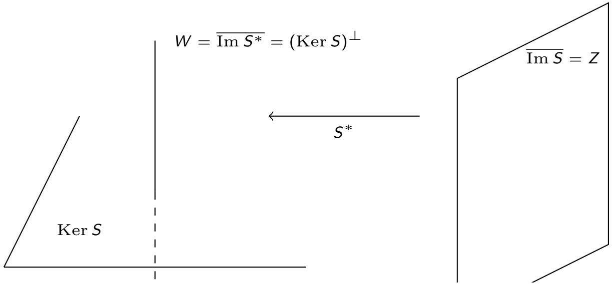
- The adjoint S^* of S is an operator from Z to X and satisfies

$$Z = \overline{\mathrm{Im}\, S} = (\mathrm{Ker}\, S^*)^\perp.$$

So $\mathrm{Ker}\, S^* = \{0\}$, i.e. S^* is injective.

Application 2: closed operators

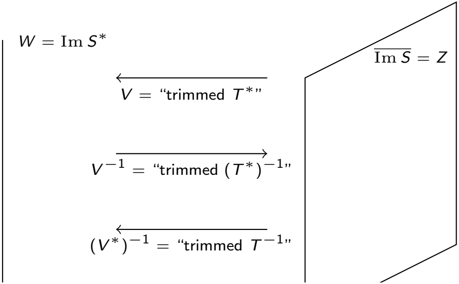
Proof:



- Note that if P denote the orthogonal projection of Y onto Z, then $S = PT$ and so $S^* = T^*P^* = T^*|_Z$. Since $T^*|_{Z^\perp} = 0$, this implies that $\mathrm{Im}\, S^* = W$ (check this!).
- We now use the same trick to rid of the cokernel of S^* (which is the kernel of S): we define $V \in \mathscr{B}(Z, W)$ by $Vz = S^*z$ so that V is a bijection and has a bounded inverse, in view of the inverse mapping theorem.

Application 2: closed operators

Proof:



- Therefore V^* is invertible and $(V^*)^{-1} = (V^{-1})^* \in \mathscr{B}(Z, W)$.
- To conclude, we show that $T \circ (V^*)^{-1} = I_Z$, as this would imply that $\mathrm{Im}\, T = Z$ which is closed.
- We follow our nose: With $z \in Z$ and $w = (V^*)^{-1} z \in W$, we compute, for $y \in Y$:

$$\langle Tw, y\rangle_Y = \langle Sw, y\rangle_Y = \langle w, S^* y\rangle_X = \langle w, Vy\rangle_X = \langle V^* w, y\rangle_Y = \langle z, y\rangle_Y.$$

Therefore $Tw = z$ and so $T \circ (V^*)^{-1} = I_Z$ as desired.

Application 3

Example

Let X be a Banach space and $(x_n) \subset X$ and $(x_n^*) \subset X^*$ are such that

- $x_n^*(x_m) = \delta_{mn}$, and
- for all $x \in X$, $(x_n^*(x)) \in c_0$.

Show that the map $x \mapsto (x_n^*(x))$ defines a bounded linear operator T from X into c_0. Show further that if T is bijective then the sequence (S_n) with $S_n = x_1 + \ldots + x_n$ is bounded.

Sketch

- A related problem occurred in the 2019 exam where X was the space of continuous 2π-periodic function and $x_n = e^{inx}$.

Application 3

Sketch

- An application of the principle of uniform boundedness gives that (x_n^*) is bounded in X^*.
- This implies $T \in \mathscr{B}(X, c_0)$.
- If T is bijective, then T has a bounded inverse. Thus, with $C = \|T^{-1}\|$, we have

$$\|x\| = \|T^{-1}Tx\| \leq C\|Tx\|_\infty.$$

- Now recall that $S_n = x_1 + \ldots + x_n$. This gives $TS_n = (\underbrace{1, \ldots, 1}_{n \text{ terms}}, 0, \ldots)$ and so $\|TS_n\|_\infty = 1$. The last assertion follows.

Counterexamples

Example

Let $X = \ell^2$ and $T \in \mathscr{B}(X)$ be given by

$$T(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots).$$

Let $Y = \mathrm{Im}\, T$. Show that Y is a proper dense subspace of X and T regarded as a bijective map between X and Y is not open.

Sketch

- It is easy to check that T is self-adjoint and injective. It follows that $Y^\perp = \mathrm{Ker}\, T^* = 0$ and so Y is dense in X.
- The sequence $(1, 1/2, 1/3, \ldots)$ belongs to ℓ^2 but Y, so Y is a proper dense subspace of X.
- Let $S \in \mathscr{B}(X, Y)$ be defined by $Sx = Tx$ and suppose that S is open. As S is a bijection, this implies that its algebraic inverse is continuous, hence bounded.

Counterexamples

Sketch

- $S^{-1} \in \mathscr{B}(Y, X)$.
- As Y is dense in X, S^{-1} has a unique extension $R \in \mathscr{B}(X)$.
- Now if $y \in Y$, we have $TRy = TS^{-1}y = y$. Therefore, by continuity $TRx = x$ for all $x \in X$. This is a contradiction as $\operatorname{Im} T = Y \neq X$.

Counterexamples

Example

There exist an incomplete normed vector space X, a Banach space Y and an operator $T \in \mathscr{B}(X, Y)$ which is surjective and but is not open.

We use without proof the following fact:

Fact

Let $(Y, \|\cdot\|)$ be an infinite dimensional Banach space. There exists a norm $\|\cdot\|_1$ on Y such that $(Y, \|\cdot\|_1)$ is incomplete and $\|y\|_1 \geq \|y\|$ for all $y \in Y$.

- The proof of the fact uses the axiom of choice.

Counterexamples

- We let $X = Y$, equipped with $\|\cdot\|_1$.
- Then the identity map $I : (X, \|\cdot\|_1) \to (Y, \|\cdot\|)$ is bounded and surjective.
- If I was open, we would have that the inverse of I is bounded, which would imply that $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent. This is impossible as $(X, \|\cdot\|_1)$ is incomplete.