B4.2 Functional Analysis II
Lecture 2

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In the last lecture

- Definition and examples of Hilbert spaces.
- Cauchy-Schwarz' inequality.
- Orthogonality in Hilbert spaces.
- Projection to closed convex sets in Hilbert spaces.

In this lecture

- The projection theorem
- Orthonormal sets
- Pythagoras' theorem
- Bessel's inequality

The projection theorem

- In the last lecture, we exhibited an example of of a direct sum decomposition $X = Y \oplus Y^\perp$ where $X = L^2(0,1)$, Y is the space of constant functions and $Y^\perp$ is the subspace of L^2 functions with zero average.
- We will now see that this is true in general, provided that Y is closed.

The projection theorem

Theorem (Projection theorem)

If Y is a closed subspace of a Hilbert space X, then Y and $Y^\perp$ are complementary subspaces: $X = Y \oplus Y^\perp$, i.e. every $x \in X$ can be decomposed uniquely as a sum of a vector in Y and in $Y^\perp$.

Proof

- First observe that if $x \in Y \cap Y^\perp$ then x is perpendicular to itself, i.e. $\langle x, x\rangle = 0$ and so $x = 0$. Hence $Y \cap Y^\perp = \{0\}$.
- It remains to show that $X = Y + Y^\perp$.
  - ⋆ Take an arbitrary $x \in X$. We need to write x as the sum of an element in Y and an element of $Y^\perp$.
  - ⋆ Note that Y is a non-empty closed convex subset of X. Hence there is a point $y_0 \in Y$ which is closer to x than any other points in Y.

    We will be done if we can show that $x - y_0 \in Y^\perp$.

The projection theorem

Proof

- $Y \cap Y^\perp = \{0\}$.
- It remains to show that $X = Y + Y^\perp$.
  - ...we will be done if we can show that $x - y_0 \in Y^\perp$.
  - Take $y \in Y$ and $t \in \mathbb{R}$, we have

$$\|x - y_0\|^2 \leq \|x - \underbrace{(y_0 - ty)}_{\in Y}\|^2 = \|x - y_0\|^2 + 2t \,\mathrm{Re}\,\langle x - y_0, y\rangle + t^2 \|y\|^2.$$

  - This is the case only if $\mathrm{Re}\,\langle x - y_0, y\rangle = 0$. This concludes the proof if the scalar field is real.
  - If the scalar field is complex, we apply the above to iy to get $\mathrm{Re}\,\langle x - y_0, iy\rangle = 0$, which gives $\mathrm{Im}\,\langle x - y_0, y\rangle = 0$. We thus have $x - y_0 \perp y$, and so $x - y_0 \in Y^\perp$ as wanted.

The projection theorem

Remark

- *The theorem is wrong if Y is not closed. For example, if Y is a proper dense subspace of X (e.g. $X = L^2(0,1)$ and $Y = C[0,1]$), then $Y^\perp = 0$ and clearly $X \neq Y \oplus Y^\perp$.*
- *Applying the theorem to Y and to $Y^\perp$, we have $Y = Y^{\perp\perp}$ if Y is closed.*
- *The theorem implies that every closed subspace Y of a Hilbert space X has a closed complement Z such that $X = Y \oplus Z$. This is not true for all Banach spaces.*

Theorem (Lindenstrauss-Tzafriri)

Let $(X, \|\cdot\|)$ be a Banach space. If every closed subspace of X admits a closed complement in X, then $(X, \|\cdot\|)$ is isomorphic to a Hilbert space.

Projection vs. Complement space

Theorem

Let X be a Banach space and Y a closed subspace. Then Y admits a complement space in X if and only if there exists a projection $P : X \to Y$, i.e. a bounded linear map with the property $Py = y$ for all $y \in Y$.

Proof (modulo a result proven later)

- $(\Rightarrow)$ Suppose Y has a complement space Z in X so that $X = Y \oplus Z$. Then the map P is given by $Px = y$ whenever $x = y + z$ with $y \in Y$ and $z \in Z$. P is well-defined in view of the direct sume.
- It is clear that P is linear and $Py = y$ for all $y \in Y$.
- The boundedness of P follows from the closed graph theorem which we will learn later in the course.

Projection vs. Complement space

Sketch of proof

- ($\Leftarrow$) Conversely, let $P : X \to Y$ be bounded linear such that $Py = y$ for all $y \in Y$.
- Let $Z = Ker\, P$, which is a closed subspace of X. We claim that $X = Y \oplus Z$.
- Note that for every $x \in X$, $Px \in Y$ so $P^2 x = P(Px) = Px$, i.e. $(I - P)x \in Z$. Therefore every $x \in X$ can be written as $x = Px + (I - P)x \in Y + Z$.
- Next, if $x \in Y \cap Z$, then $x = Px$ because $x \in Y$ and $Px = 0$ because $x \in Z = Ker\, P$. So $x = 0$.
- We conclude that $X = Y \oplus Z$ as claimed.

$H^1(-\pi,\pi)$ in $L^1(-\pi,\pi)$

Theorem

The Hardy space $H^1(-\pi,\pi)$ of complex-valued integrable functions whose Fourier series have the form $\sum_{n\geq 0} a_n e^{inx}$ is a closed subspace of $L^1(-\pi,\pi)$ which has no complement space.

Elements of proof

- Suppose by contradiction that $Y = H^1(-\pi,\pi)$ has a complement space in $X = L^1(-\pi,\pi)$. Then there exists a projection $P : X \to Y$.
- (Hard) It can be show that such map must satisfies

$$P\Big(\sum_{n=-\infty}^{\infty} a_n e^{inx}\Big) = \sum_{n\geq 0} a_n e^{inx}.$$

$H^1(-\pi,\pi)$ in $L^1(-\pi,\pi)$

Elements of proof

- ...there exists a projection $P: X \to Y$ and
  $$P\Big(\sum_{n=-\infty}^{\infty} a_n e^{inx}\Big) = \sum_{n\geq 0} a_n e^{inx}.$$
- Consider
  $$f_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} \text{ for } 0 < r < 1.$$
- Explicit summation gives $f_r \geq 0$ and $\|f_r\| = 1$.
- We also have $P(f_r) = \frac{1}{1-re^{i\theta}} \to \frac{1}{1-e^{i\theta}}$ a.e. as $r \to 1$. Therefore, by Fatou's lemma,
  $$\liminf_{r\to 1} \|P(f_r)\| \geq \int_{-\pi}^{\pi} \frac{1}{|1-e^{i\theta}|}\,d\theta = \infty.$$

  This contradicts the assumption that P was bounded.

The span and the closed linear span of a set

Definition

Let S be a subset in a Hilbert space X.

- The span of S, denoted by $Span(S)$, is the set of all finite linear combinations of elements in S.
- The closed linear span of a set S, denote by $\overline{Span(S)}$, is the smallest closed linear subspace of X containing S, i.e. the intersection of all such subspaces.

Facts:

$$\text{The closed linear span of } S = \text{The closure of } Span(S) = S^{\perp\perp}.$$

Orthonormal sets and bases

Definition

Let S be a subset of a Hilbert space X.

- S is called an orthonormal set if $\|x\| = 1$ for all $x \in S$ and $\langle x, y \rangle = 0$ for all $x \neq y \in S$.
- S is called an orthonormal basis (or a complete orthonormal set) for X if S is an orthonormal set and its closed linear span is X.

Theorem

Every Hilbert space contains an orthonormal basis.

We will give the proof for the case of separable Hilbert spaces.

Existence of orthonormal bases

Proof for separable Hilbert spaces

- Let X be a separable Hilbert space with an at most countable dense subset $S = \{y_1, y_2, \ldots\}$.
- We construct an orthonormal sequence $\{x_1, x_2, \ldots\}$ inductively using a Gram-Schmidt type process as follows.
  - ⋆ If $y_1 \neq 0$, let $x_1 = \frac{1}{\|y_1\|} y_1$ and let $i_1 = 1$. If $y_1 = 0$, then $y_2 \neq 0$ and we let $x_1 = \frac{1}{\|y_2\|} y_2$ and $i_1 = 2$. Note that $E_1 := Span(\{x_1\})$ is equal to $Span(\{y_1\})$ in the former case and to $Span(\{y_1, y_2\})$ in the latter case.
  - ⋆ Suppose we have constructed the orthogonal sequence $\{x_1, \ldots, x_n\}$ and the index i_n such that $E_n := Span(\{x_1, \ldots, x_n\}) = Span(\{y_1, \ldots, y_{i_n}\})$.
  - ⋆ If $S \subset E_n$, we deduce that $X = E_n$ and we are done.
  - ⋆ Otherwise, we let i_{n+1} be the smallest index such that $y_{i_{n+1}} \notin E_n$ and correct $\{x_1, \ldots, x_n, y_{i_{n+1}}\}$ to an orthonormal sequence $\{x_1, \ldots, x_n, x_{n+1}\}$.

Existence of orthonormal bases

Proof for separable Hilbert spaces

- We have thus constructed an orthonormal sequence $\tilde{S} = \{x_1, x_2, \ldots\}$ of X and the indices $i_1 < i_2 < \ldots$ such that

$$E_n = Span(\{x_1, \ldots, x_n\}) = Span(\{y_1, \ldots, y_{i_n}\}).$$

- This implies that $Span(\tilde{S}) = Span(S)$. Taking closure we have $\overline{Span(\tilde{S})} = \overline{Span(S)} = X$.

Example of orthonormal bases

- Let $X = \ell^2$. The standard basis $S = \{e_1, e_2, \ldots\}$ with $e_n = (\delta_{mn})_{m=1}^\infty$ is an orthonormal basis for X.
- Let $X = L^2(-\pi, \pi)$. We will show later in the course that the trigonometric system $S = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \ldots\}$ is an orthonormal basis for X.
- Let $X = L^2(-1, 1)$. The set of polynomials is dense in X. Applying the Gram-Schmidt process to the set of monomials $\{1, x, x^2, \ldots\}$ yields an orthogonal basis $\{P_1, P_2, \ldots\}$ for X such that $\deg P_n = n$. Traditionally, one normalises $\|P_n\|^2 = \frac{2}{2n+1}$. This family is called the Legendre polynomials, which you may have run into if you took DE2. It satisfies the so-called Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

Pythagoras' theorem

Theorem (Pythagorean theorem)

Let X be a Hilbert space and $S = \{x_1, x_2, \ldots, x_m\}$ be a finite orthonormal set in X. For every $x \in X$, there holds

$$\|x\|^2 = \sum_{n=1}^{m} |\langle x, x_n\rangle|^2 + \Big\|x - \sum_{n=1}^{m} \langle x, x_n\rangle x_n\Big\|^2.$$

Proof

- Let $y = \sum_{n=1}^m \langle x, x_n\rangle x_n$. Using $\|y\|^2 = \langle y, y\rangle$ and the orthonormality of S, we have $\|y\|^2 = \sum_{n=1}^m |\langle x, x_n\rangle|^2$.
- We also have $\langle x, y\rangle = \sum_{n=1}^m |\langle x, x_n\rangle|^2$.
- The conclusion follows from the identity

$$\|x\|^2 = \|y\|^2 + 2Re\langle x - y, y\rangle + \|x - y\|^2.$$

Bessel's inequality

Lemma (Bessel's inequality)

Let X be a Hilbert space and $S = \{x_1, x_2, \dots\}$ be an orthonormal sequence in X. Then, for every $x \in X$, there holds

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq \|x\|^2.$$

Proof

- Apply Pythagoras' theorem, we have for each m that

$$\|x\|^2 = \sum_{n=1}^{m} |\langle x, x_n \rangle|^2 + \Big\| x - \sum_{n=1}^{m} \langle x, x_n \rangle x_n \Big\|^2.$$

Bessel's inequality

Proof

- $\|x\|^2 = \sum_{n=1}^{m} |\langle x, x_n\rangle|^2 + \left\| x - \sum_{n=1}^{m} \langle x, x_n\rangle x_n \right\|^2.$
- Dropping the last term, we then have

$$\sum_{n=1}^{m} |\langle x, x_n\rangle|^2 \leq \|x\|^2.$$

Since this is true for all m, the conclusion follows.

Further on Bessel's inequality

Bessel's inequality begs a number of questions:

- Is the inequality sharp? – This is easy to answer: with $x = x_1$, we have $\|x\|^2 = 1 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$.
- Is it an inequality? (You may be reminded of Parserval's identity from Prelims Introduction to PDEs and Fourier Series.)
  If so, for which x is the inequality attained? – These will be answered in the next lecture.