B4.2 Functional Analysis II
Lecture 14

Luc Nguyen
luc.nguyen@maths

University of Oxford

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In the last 3 lectures

- Convergence/Divergence of Fourier series

In this lecture

- The ABCs of spectral theory for bounded linear operators.
- Spectra of normal operators.

Basic definitions

Definition

Let X be a complex Banach space and $T \in \mathscr{B}(X)$.

- The spectrum $\sigma(T)$ of T is the set of complex numbers λ such that $\lambda I - T$ has no inverse in $\mathscr{B}(X)$.
- The resolvent set $\rho(T)$ of T is the complement of $\sigma(T)$ in $\mathbb{C}$. If $\lambda \in \rho(T)$, then $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.

Facts from FA1:

- $\sigma(T)$ is closed, non-empty and contained in $\{|\lambda| \leq \|T\|\}$.
- Gelfand's formula: The spectral radius is

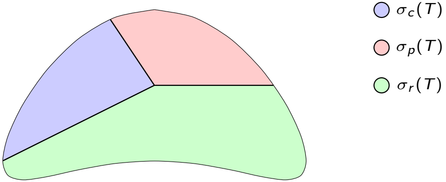
$$\mathrm{rad}(\sigma(T)) := \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n \to \infty} \|T^n\|^{1/n} = \inf_n \|T^n\|^{1/n}.$$

Refined definitions

Definition

- Point spectrum: $\lambda \in \sigma_p(T)$ if $\lambda I - T$ is not injective.
  λ is called an eigenvalue of T and the non-trivial elements of $\mathrm{Ker}(\lambda I - T)$ are called the eigenvectors of T.
- Residual spectrum: $\lambda \in \sigma_r(T)$ if $\lambda I - T$ is injective and its range $\mathrm{Im}(\lambda I - T)$ is not dense in X.
- Continuous spectrum: $\lambda \in \sigma_c(T)$ if $\lambda I - T$ is injective and its range $\mathrm{Im}(\lambda I - T)$ is a proper dense subset of X.
- Approximate point spectrum: $\lambda \in \sigma_{ap}(T)$ if there is a sequence $(x_n) \subset X$ such that $\|x_n\| = 1$ and $\|Tx_n - \lambda x_n\| \to 0$.
  λ is called an approximate eigenvalue of T.

A schematic diagram



The spectrum of a bounded linear operator.

- $\sigma(T)$ is split as $\sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$.
- $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are mutually disjoint.

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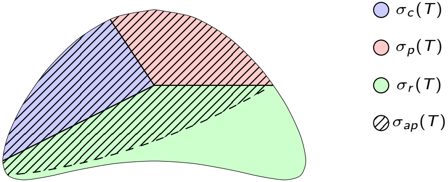

The spectrum of a bounded linear operator.

- $\sigma(T)$ is split as $\sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$.
- $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are mutually disjoint.
- $\sigma_p(T)$ is a subset of $\sigma_{ap}(T)$.
- We will now see that $\sigma_c(T)$ is also a subset of $\sigma_{ap}(T)$.

$\sigma_c(T) \subset \sigma_{ap}(T)$

Lemma

Let $T \in \mathscr{B}(X)$ be a bounded linear operator on a Banach space X. Then $\sigma_c(T) \subset \sigma_{ap}(T)$.

Proof

- Suppose by contradiction that there exists $\lambda \in \sigma_c(T) \setminus \sigma_{ap}(T)$.
- $\lambda \in \sigma_c(T)$ means that $\lambda I - T$ is injective and its range $Y = \mathrm{Im}(\lambda I - T)$ is dense in X. In particular, $\lambda I - T$ considered as a map from X into Y is bijective and has an inverse, say $U : Y \to X$.
- $\lambda \notin \sigma_{ap}(T)$ means that there is some $c > 0$ such that

$$\|(\lambda I - T)x\| \geq c \text{ for all } x \in X, \|x\| = 1.$$

Equivalently, this means $\|(\lambda I - T)x\| \geq c\|x\|$ for all $x \in X$.

$\sigma_c(T) \subset \sigma_{ap}(T)$

- $\lambda \in \sigma_c(T) \setminus \sigma_{ap}(T)$.
- $Y = \mathrm{Im}(\lambda I - T)$ is dense in X and $\lambda I - T$ considered as a map from X into Y has an inverse $U : Y \to X$.
- $\|(\lambda I - T)x\| \geq c\|x\|$ for all $x \in X$.
- This means that $\|y\| \geq c\|Uy\|$ for all $y \in Y$, i.e. U is bounded.
- Since Y is dense in X, U extends to $\bar{U} \in \mathscr{B}(X)$.
- We are now in position to deduce a contradiction: If $p \in X \setminus Y$ and $p_n \in Y$ such that $p_n \to p$, then $Up_n \to \bar{U}p$ and so

$$(\lambda I - T)\bar{U}p = \lim_{n\to\infty}(\lambda I - T)Up_n = \lim_{n\to\infty} p_n = p.$$

This shows that p belongs to Y, a contradiction.

$\sigma_r(T) \subset \sigma_p(T')$

Lemma

Let $T \in \mathscr{B}(X)$ be a bounded linear operator on a Banach space X. Then $\sigma_r(T) \subset \sigma_p(T')$ where $T' \in \mathscr{B}(X^*)$ is the dual of T.

Proof

- Suppose $\lambda \in \sigma_r(T)$ so that $Y = \mathrm{Im}(\lambda I - T)$ is a proper non-dense subspace of X.
- By the Hahn-Banach theorem, there is an element $\ell \in X^*$, $\|\ell\|_* = 1$ such that $\ell(\bar{Y}) = 0$.
- Now, for all $x \in X$ we have

$$\big((\lambda I - T')\ell\big)(x) = \ell\big((\lambda I - T)x\big) = 0.$$

This means $(\lambda I - T')\ell = 0$, i.e. λ is an eigenvalue of T'.

Example

Example

Let X be a complex Banach space and $T \in \mathscr{B}(X)$. Show that if λ is on the (topological) boundary of $\sigma(T)$, then $\lambda \in \sigma_{ap}(T)$.

- Suppose by contradiction that $\lambda \notin \sigma_{ap}(T)$. As seen earlier, this means that there is some $c > 0$ such that
$$\|(\lambda I - T)x\| \geq c\|x\| \text{ for all } x \in X.$$
- Since λ is on the boundary of $\sigma(T)$, there exists $\lambda_n \in \rho(T)$ such that $\lambda_n \to \lambda$. By the above, we have for large n that
$$\|(\lambda_n I - T)x\| \geq \frac{1}{2}c\|x\| \text{ for all } x \in X.$$
This gives $\|R_{\lambda_n}(T)\| \leq \frac{2}{c}$ for large n.

Examples

- $\|R_{\lambda_n}(T)\| \leq \frac{2}{c}$ for large n.
- Now

$$\lambda I - T = (\lambda_n I - T) + (\lambda - \lambda_n)I = (\lambda_n I - T)\Big(I + \underbrace{(\lambda - \lambda_n)R_{\lambda_n}(T)}_{=: S_n}\Big).$$

By the above estimate, $\|S_n\| < 1$ for large n. This implies that $I + S_n$ and hence $\lambda I - T$ are invertible, contradicting the fact that $\lambda \in \sigma(T)$.

Remark

It can also be shown that if λ is on the boundary of $\sigma(T)$, then $Im(\lambda I - T) \neq X$.

Hilbert settings

Let X be a (complex) Hilbert space and $T \in \mathscr{B}(X)$.
Note: $(\lambda I - T)^* = \bar{\lambda} I - T^*$.

	T	T^*
$\lambda \in \sigma_p(T)$	$\mathrm{Ker}(\lambda I - T) \neq 0$	$\mathrm{Im}(\bar{\lambda} I - T^*)$ isn't dense $(\Rightarrow \bar{\lambda} \in \sigma_p(T^*) \cup \sigma_r(T^*))$
$\lambda \in \sigma_c(T)$ $\Leftrightarrow \bar{\lambda} \in \sigma_c(T^*)$	$\mathrm{Ker}(\lambda I - T) = 0$, $\mathrm{Im}(\lambda I - T)$ is dense	$\mathrm{Ker}(\bar{\lambda} I - T^*) = 0$, $\mathrm{Im}(\bar{\lambda} I - T^*)$ is dense
$\lambda \in \sigma_r(T)$	$\mathrm{Ker}(\lambda I - T) = 0$, $\mathrm{Im}(\lambda I - T)$ isn't dense	$\mathrm{Ker}(\bar{\lambda} I - T^*) \neq 0$ $(\Rightarrow \bar{\lambda} \in \sigma_p(T^*))$

$\sigma(T) = \sigma_{ap}(T) \cup \sigma'_p(T^*) = \sigma'(T^*)$

In particular, we have

Theorem

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$. Then

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T') = \sigma_{ap}(T) \cup \sigma'_p(T^*) = \sigma'(T^*)$$

where $\sigma'_p(T^*) = \{\lambda : \bar{\lambda} \in \sigma_p(T^*)\}$ *and* $\sigma'(T^*) = \{\lambda : \bar{\lambda} \in \sigma(T^*)\}$.

Example

Example

Let X be a (complex) Hilbert space and $T \in \mathscr{B}(X)$. Show that if $\lambda \in \sigma(T)$, then there exists $(x_n) \subset X$, $\|x_n\| = 1$ such that $\langle Tx_n, x_n \rangle \to \lambda$. In other words,

$$\sigma(T) \subset \overline{\{\langle Tx, x \rangle : \|x\| = 1\}} \subset \mathbb{C}.$$

- If $\lambda \in \sigma_{ap}(T)$, then we can select (x_n) with $\|x_n\| = 1$ such that $(\lambda I - T)x_n \to 0$. By Cauchy-Schwarz' inequality, this implies

$$\lambda - \langle Tx_n, x_n \rangle = \langle \lambda x_n - Tx_n, x_n \rangle \to 0$$

  and so $\langle Tx_n, x_n \rangle \to \lambda$.
- If $\lambda \notin \sigma_{ap}(T)$, then $\bar{\lambda} \in \sigma_p(T^*)$ and so there is an x with $\|x\| = 1$ and $T^* x = \bar{\lambda} x$. This implies with $x_n = x$ that

$$\langle Tx_n, x_n \rangle = \langle Tx, x \rangle = \langle x, T^* x \rangle = \langle x, \bar{\lambda} x \rangle = \lambda.$$

Normal operators

Definition

Let X be a complex Hilbert space. An operator $T \in \mathscr{B}(X)$ is called normal if $TT^* = T^*T$.

Proposition

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$. T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in X$.

Proof

- $(\Rightarrow)$ This is straightforward:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = \langle x, TT^*x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2.$$

Normal operators

Proof

- ($\Leftarrow$) Suppose that $\|Tx\| = \|T^*x\|$ for all $x \in X$.
  By polarisation, we have $\langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle$ for all $x, y \in X$.
  Now we reverse the argument we did above

$$\langle x, T^*Ty\rangle = \langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle = \langle x, TT^*y\rangle.$$

  This implies that $T^*T = TT^*$.

Normal operators

Corollary

Let X be a complex Hilbert space. If $T \in \mathscr{B}(X)$ is normal then $Ker\, T = Ker\, T^*$.

Proof: This is because $\|Tx\| = \|T^*x\|$.

Spectra of normal operators

Corollary

Let X be a complex Hilbert space. If $T \in \mathscr{B}(X)$ is normal then $\sigma_r(T) = \emptyset$ and $\sigma(T) = \sigma_{ap}(T)$.

Proof

- It suffices to show that $\sigma_r(T)$ is empty.
- We knew that $\sigma_r(T) \subset \sigma_p(T') = \sigma_p'(T^*)$.
- Now, note that $\lambda I - T$ is also normal. By the previous corollary, we have that $\bar{\lambda} I - T^*$ is injective if and only if $\lambda I - T$ is injective, i.e. $\sigma_p'(T^*) = \sigma_p(T)$.
- So $\sigma_r(T) \subset \sigma_p(T)$. As these two sets are disjoint, this is possible only if $\sigma_r(T)$ is empty.

Orthogonality of eigenvectors of normal operators

Proposition

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be normal. If x and y are eigenvectors of T corresponding to different eigenvalues, then $\langle x, y \rangle = 0$.

Proof

- Suppose $Tx = \lambda x$ and $Ty = \tilde{\lambda} y$. We have
$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Tx, y \rangle = \langle x, T^* y \rangle.$$
- As $\tilde{\lambda} I - T$ is normal, we have $0 = \|(\tilde{\lambda} I - T)y\| = \|(\bar{\tilde{\lambda}} I - T^*)y\|$ and so $T^* y = \bar{\tilde{\lambda}} y$.
- It follows that
$$\lambda \langle x, y \rangle = \langle x, T^* y \rangle = \langle x, \bar{\tilde{\lambda}} y \rangle = \tilde{\lambda} \langle x, y \rangle.$$
Since $\lambda \neq \tilde{\lambda}$, this implies $\langle x, y \rangle = 0$.