B4.2 Functional Analysis II
Lecture 9

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In the last lecture

- Weak convergence: Examples and basic properties.

In this lecture

- Mazur's theorem.

Recap

Suppose that X is a normed vector space and $(x_n) \subset X$ converges weakly to x. We knew:

- (x_n) is bounded.
- $\|x\| \leq \liminf \|x_n\|$.

In particular, if $x_n \in \overline{B(0, R)}$, then $x \in \overline{B(0, R)}$.

Mazur's theorem

Theorem (Mazur's theorem)

Let K be a closed convex subset of a normed vector space X, (x_n) be a sequence of points in K converging weakly to x. Then $x \in K$.

Corollary

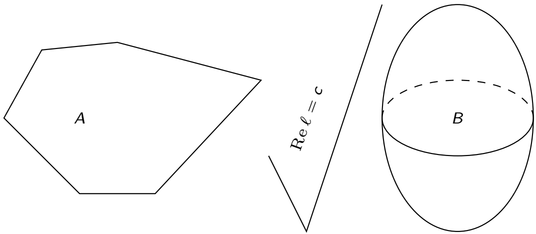
The weak limit x belong to the closure of the convex hull of $S = \{x_1, x_2, \dots\}$. In other words, a sequence of finite linear convex combinations of the x_n's converges strongly to x.

Extended hyperplane separation theorem

Theorem (Extended hyperplane separation theorem)

Let X be a normed vector space, A and B be disjoint convex subsets of X. Suppose that at least one of them has an interior point. Then A and B can be separated by a hyperplane, i.e. there is a non-zero linear function ℓ and a number c such that

$$\mathrm{Re}\,\ell(a) \le c \le \mathrm{Re}\,\ell(b) \text{ for all } a \in A, b \in B.$$



Extended hyperplane separation theorem

Remark

The linear function ℓ is in fact bounded.

Proof

- Since one of the set, say B, has non-empty interior, we can find a ball $B(x_0, r_0)$ such that $\mathrm{Re}\,\ell(x) \geq c$ for $x \in B(x_0, r_0)$.
- It follows that for every $z \in B(0, 1)$,

$$\mathrm{Re}\,\ell(z) = \frac{1}{r_0}(\mathrm{Re}\,\ell(x_0 + rz) - \mathrm{Re}\,\ell(x_0)) \geq \frac{1}{r_0}(c - \mathrm{Re}\,\ell(x_0)) =: -M.$$

- Using $-z$ in place of z, we have $-\mathrm{Re}\,\ell(z) \geq -M$, and so

$$|\mathrm{Re}\,\ell(z)| \leq M.$$

This proves the boundedness of ℓ when the field is real. The complex case is dealt with by using iz in the above.

Proof of Mazur's theorem

- Let $K \subset X$ be closed and convex and suppose by contradiction that $(x_n) \subset K$ converges weakly to some $x \notin K$.
- Since K is closed, its complement is open. Hence there is some $r > 0$ such that $B(x, r) \cap K = \emptyset$.
- Applying the extended hyperplane separation theorem to the set K and $\overline{B(x, r)}$, we find a bounded linear functional $\ell_0 \in X^*$, $\ell_0 \neq 0$ and a number $c \in \mathbb{R}$ such that

$$\mathrm{Re}\,\ell_0(a) \leq c \leq \mathrm{Re}\,\ell_0(b) \text{ for all } a \in K \text{ and } b \in B(x, r).$$

- Taking $a = x_n$ gives $\mathrm{Re}\,\ell_0(x_n) \leq c$. As $x_n \rightharpoonup x$, we have $\mathrm{Re}\,\ell_0(x) \leq c$.
- So we have $\mathrm{Re}\,\ell_0(x) \leq \mathrm{Re}\,\ell_0(z)$ for all $z \in B(x, r)$, and so, for $w \in B(0, 1)$,

$$\mathrm{Re}\,\ell_0(w) = \frac{1}{r}(\mathrm{Re}\,\ell_0(x + rw) - \mathrm{Re}\,\ell_0(x)) \geq 0.$$

As seen on the previous slide, this implies $\ell_0 = 0$, which is a

Banach-Saks' theorem

By the corollary to Mazur's theorem, if $x_n \rightharpoonup x$ then a finite convex linear combination of the x_n's converges strongly to x. In Hilbert spaces, this can be improved substantially:

Theorem (Banach-Saks)

Let X be a Hilbert space. Then every weakly convergent sequence (x_m) in X has a subsequence (x_{m_k}) which converges in the Cesaro sense , i.e.

$$\frac{1}{n}\sum_{k=1}^{n} x_{m_k} \text{ converges as } j \to \infty.$$

A difficult result of Kakutaki asserts that the conclusion hold if X is a uniformly convex Banach space.

Banach-Saks' thereom

Sketch of proof:

- WLOG, we assume that $x_m \rightharpoonup 0$ and $\|x_m\| \leq 1$.
- Claim: There is a sequence $m_1 = 1 < m_2 < \ldots$ such that

$$\|x_{m_1} + \ldots + x_{m_n}\|^2 \leq 3n.$$

  This can be done by induction. For the inductive step, you will need to select $x_{m_{n+1}}$ such that $|\langle x_{m_1} + \ldots + x_{m_n}, x_{m_{n+1}}\rangle| \leq 1$, which is attainable as $\langle a, x_m\rangle \to 0$ for all $a \in X$.
- But then we have

$$\|\frac{1}{n}(x_{m_1} + \ldots + x_{m_n})\|^2 \leq \frac{3}{n} \to 0.$$

Example 1

Example

Let $X = L^p(-\pi,\pi)$, $1 < p < \infty$ and $x_n(t) = \sin nt$. Determine if (x_n) converges strongly or weakly and, if so, identify its limit.

- If $p = 2$, X is a Hilbert space, and (x_n) is an orthogonal sequence. Hence (x_n) does not converge strongly and converge weakly to zero (by Bessel's inequality).
- For $p \neq 2$, one can show that (x_n) doesn't converge strongly by doing a direct computation to show that it is not Cauchy. But this is messy.
- We claim $x_n \rightharpoonup 0$, i.e. $\int_{-\pi}^{\pi} \sin nt\, g(t)\, dt \to 0$ for all $g \in L^{p'}(-\pi,\pi) \cong X^*$. We have seen this before in Lecture 5. It suffices to check the convergence for g being the characteristic function of an open interval, which is straightforward.

Example 1

- If (x_n) converges strongly, its strong limit must be the same as its weak limit, which is zero. But the sequence (x_n) have constant positive norm:

$$\|x_n\|^p = \int_{-\pi}^{\pi} |\sin nt|^p \, dt = \frac{2}{n} \int_0^{n\pi} |\sin s|^p \, ds$$
$$= 2 \int_0^{\pi} |\sin s|^p \, ds \not\to 0!$$

So (x_n) does not converge strongly.

Example 2

Example

Let $X = L^1(\mathbb{R}^n)$. Let $E_1, E_2, \ldots$ are disjoint measurable subsets of finite positive measure of $\mathbb{R}^n$, and $f_k = \frac{1}{|E_k|}\chi_{E_k}$. Determine if (f_k) converges strongly or weakly and, if so, identify its limit.

- It is easy to see that $\|f_k\| = 1$ and $\|f_k - f_m\| = 2$ if $k \neq m$. So (f_k) is not Cauchy and hence not strongly convergent.
- If all E_k has measure 1, the sequence (f_k) is actually orthonormal in $L^2(\mathbb{R}^n)$ and so converges weakly to 0 in $L^2(\mathbb{R}^n)$. One may therefore be tempted to say that (f_k) converges weakly to 0 in $L^1(\mathbb{R}^n)$. This isn't true!
- We claim that (f_k) doesn't converge weakly.
- Suppose by contradiction that $f_k \rightharpoonup f$, i.e.

$$\int_{\mathbb{R}^n} f_k g \to \int_{\mathbb{R}^n} fg \text{ for all } g \in L^\infty(\mathbb{R}^n) \cong (L^1(\mathbb{R}^n))^*.$$

Example 2

- ... Suppose by contradiction that $\int_{\mathbb{R}^n} f_k g \to \int_{\mathbb{R}^n} fg$ for all $g \in L^\infty(\mathbb{R}^n)$.
- Using $g = sign(f)\chi_{E_1}$, $g = sign(f)\chi_{E_2}$, ..., we obtain $\int_{E_m} |f| = 0$, i.e. $f = 0$ a.e. in $\cup E_k$.
- Similarly, using $g = sign(f)\chi_{\mathbb{R}^n \setminus (\cup E_k)}$, we have $f = 0$ a.e. in $\mathbb{R}^n \setminus (\cup E_k)$. So $f = 0$.
- On the other hand, by Mazur's theorem, there is a sequence (h_k), each of which is a finite convex linear combination of f_k's, which converges to f strongly.
- To reach a contradiction, we show that $\|h\| = 1$ for any finite convex linear combination h of f_k's. Indeed, let $h = \sum_{m=1}^N c_m f_m$ with $0 \leq c_m \leq 1$ and $\sum_{m=1}^N c_m = 1$. Then

$$\|h\| = \sum_{m=1}^N \int_{E_m} |h| = \sum_{m=1}^N \int_{E_m} c_m |f_m| = \sum_{m=1}^N c_m = 1.$$

Summary

- We have exhibited a sequence with constant and positive norm in L^p which converges weakly to 0 for $1 < p < \infty$.
- We have exhibited a sequence with constant norm in L^1 which does not converge weakly.