B4.2 Functional Analysis II
Lecture 5

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HT 2021

In the last 4 lectures

- Hilbert spaces and operators on Hilbert spaces.

In this lecture

- The Baire category theorem.
- The principle of uniform boundedness.

The Baire category theorem

Definition

Let S be a subset of a metric space M.

1. We say that S is dense in M if $\bar{S} = M$.
2. We say that S is nowhere dense in M if $\bar{S}$ has empty interior.

Theorem (The Baire category theorem)

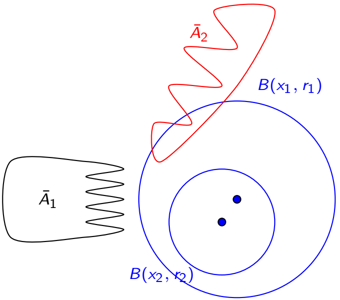
A non-empty complete metric space is never the union of a countable number of nowhere dense sets.

Proof

- Suppose that M is a complete metric space and $A_1, A_2, \ldots$ is a sequence of nowhere dense subsets of M.
  We need to show that the complement of $\cup_{n=1}^{\infty} A_n$ is non-empty.

The Baire category theorem

- To that end, we construct a Cauchy sequence (x_n) such that for every m, the tail $(x_n)_{n>m}$ lies outside of $\bar{A}_m$ in such a way that the limit of this sequence does not lie in any A_m.
- We start with A_1. Since $\bar{A}_1$ has empty interior, $\bar{A}_1 \neq M$.
  So $M \setminus \bar{A}_1$ is a non-empty open set.
  Pick a *closed* ball $\overline{B(x_1, r_1)} \subset M \setminus \bar{A}_1$ with $0 < r_1 < 1$.
- We move on with A_2. Since $\bar{A}_2$ has empty interior, $\bar{A}_2 \not\supset B(x_1, r_1)$.
  So $B(x_1, r_1) \setminus \bar{A}_2$ is a non-empty open set.
  Pick a *closed* ball $\overline{B(x_2, r_2)} \subset B(x_1, r_1) \setminus \bar{A}_2$ with $0 < r_2 < 1/2$.



The Baire category theorem

- Proceeding in this way, we obtain a sequence of balls $B(x_1, r_1)$, $B(x_2, r_2)$, ... such that $r_n < \frac{1}{2^{n-1}}$ and

$$\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \setminus \bar{A}_n.$$

- The sequence (x_n) is Cauchy: If $n, m \geq N$, then $x_n, x_m \in B(x_N, r_N)$ and so $d(x_m, x_n) \leq 2\, r_N \to 0$.
- Since M is complete, (x_n) converges to some $x \in M$.
- Since $x_n \in B(x_N, r_N)$ for $n \geq N$, we have $x \in \overline{B(x_N, r_N)} \subset B(x_{N-1}, r_{N-1}) \setminus \bar{A}_N$.
- In particular, $x \notin A_N$ for any N.
- We conclude that M is not the union of the A_n's.

The Baire category theorem – other forms

Here're some equivalent forms of the Baire category theorem:

Theorem

In a non-empty complete metric space, the following statements hold:

1. (i) *The intersection of countably many dense open sets is non-empty.*
2. (ii) *The union of countably many nowhere dense sets has empty interior.*
3. (iii) *The intersection of countably many dense open sets is dense.*

Proof

- Statement (i) is equivalent to the form stated earlier by De Morgan's law. (Check this!)
- Likewise (ii) $\Leftrightarrow$ (iii).

The Baire category theorem – other forms

Proof

- On the surface, (ii) looks stronger than the original statement. It is in fact equivalent.
- Indeed, suppose that $A_1, A_2, \ldots$ are nowhere dense, and suppose that their union contains a ball $B(x,r)$.
- Now take a closed ball $\overline{B(x,s)} \subset B(x,r)$ and let $\tilde{M} = \overline{B(x,s)} = M \cap \overline{B(x,s)}$, and $\tilde{A}_n = A_n \cap \overline{B(x,s)}$.
- Then $\tilde{M} = \cup \tilde{A}_n$ and $\tilde{A}_n$'s are nowhere dense in $\tilde{M}$, which is a contradiction.

The principle of uniform boundedness

Theorem (Principle of uniform boundedness; Banach-Steinhaus theorem)

Let X be a Banach space and Y be a normed vector space. Let $\mathscr{F} \subset \mathscr{B}(X, Y)$. If

$$\sup\{\|Tx\|_Y : T \in \mathscr{F}\} < \infty \text{ for every individual } x \in X,$$

then $\mathscr{F}$ is bounded in $\mathscr{B}(X, Y)$, i.e.

$$\sup\{\|T\| : T \in \mathscr{F}\} < \infty.$$

Loosely speaking, it says that a family of *bounded linear* operators (from a Banach space into a normed vector space) is bounded if it is pointwise bounded. It should be clear that linearity of maps is of crucial importance in the theorem.

The principle of uniform boundedness

Proof

- The proof is an application of the Baire category theorem.
- Let $A_n = \{x \in X : \|Tx\|_Y \leq n \text{ for all } T \in \mathscr{F}\}$.
- By hypothesis, each $x \in X$ belongs to some A_n and so $X = \cup_{n=1}^{\infty} A_n$.
- By the Baire category theorem, there is some n_0 such that $A_{n_0} = \bar{A}_{n_0}$ (since the A_n's are closed) has non-empty interior. We can thus pick a ball $B(x_0, r_0) \subset A_{n_0}$.
- Now suppose that $\|z\|_X < 1$, we proceed to bound $\|Tz\|_Y$ for $T \in \mathscr{F}$.

  By triangle inequality, we have $x_0 + r_0 z \in B(x_0, r_0)$ and so, by the definition of A_{n_0},

$$\|T(x_0 + r_0 z)\|_Y \leq n_0 \text{ for all } T \in \mathscr{F}.$$

The principle of uniform boundedness

Proof

- $\|T(x_0 + r_0 z)\|_Y \leq n_0$ for all $T \in \mathscr{F}$ and $\|z\| < 1$.
- Letting $z = 0$ gives $\|T(x_0)\|_Y \leq n_0$ for all $T \in \mathscr{F}$.
- By triangle inequality again, we thus have for $T \in \mathscr{F}$ and $\|z\| < 1$ that

$$\begin{aligned}\|Tz\|_Y &= \frac{1}{r_0}\|T(x_0 + r_0 z) - Tx_0\|_Y \\ &\leq \frac{1}{r_0}(\|T(x_0 + r_0 z)\|_Y + \|Tx_0\|_Y) \leq \frac{2n_0}{r_0}.\end{aligned}$$

- $\|T\|_{\mathscr{B}(X,Y)} \leq 2n_0\, r_0^{-1}$ for all $T \in \mathscr{F}$.

Application 1

Theorem

Let X be a Hilbert space and $\mathscr{F}$ be a subset of $\mathscr{B}(X)$ such that $\sup_{T\in\mathscr{F}} |\langle Tx, y\rangle| < \infty$ for each $x, y \in X$. Then $\{\|T\| : T \in \mathscr{F}\}$ is bounded.

Remark: Compare the fact that $\|T\| = \sup\limits_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|$.

Proof

- By the principle of uniform boundedness, it suffices to show that, for each fixed $x \in X$, $\{\|Tx\| : T \in \mathscr{F}\}$ is bounded.
- Fix an $x \in X$. Define $K_{T,x} \in X^*$ by $K_{T,x}(y) = \langle y, Tx\rangle$ so that $\|Tx\| = \|K_{T,x}\|_*$.
  We thus need to show that $\{\|K_{T,x}\|_* : T \in \mathscr{F}\}$ is bounded.
- By the principle of uniform boundedness again, we need to show that, for each $y \in X$, $\{|K_{T,x}(y)| : T \in \mathscr{F}\}$ is bounded. But this is true by hypothesis.

Application 2

Example

Let X be a Hilbert space. A sequence (x_n) in X is bounded if and only if $(\langle x_n, y\rangle)$ is bounded for every $y \in X$.

- This appeared in the 2019 exam.
- We view x_n as an element of X^* by identifying with $T_n(y) = \langle y, x_n\rangle$.
- We know that $\|x_n\| = \|T_n\|_*$. So (x_n) is bounded in X if and only if (T_n) is bounded in X^*. By the principle of uniform boundedness, this holds if and only if T_n is pointwise bounded. The conclusion follows.

Application 3: approximate quadrature formulae

- An approximate quadrature formula is an approximation of the integral of a continuous function on a closed interval, say $[-1, 1]$, by a weighted average of the values of the functions at some specified points of the form
$$\int_{-1}^{1} f(t)\,dt \approx \sum_{j=1}^{n} w_j f(t_j). \qquad (*)$$
- The points t_j are called the *nodes*. The numbers w_j are called the *weights*. These are independent of f.
- The weights and the nodes are designed such that the formula (*) is exact for a certain class of functions. For example, in the so-called Gaussian quadrature rule, it is required that the formula (*) is exact for all polynomials of degree $\leq 2n - 1$.

The principle of uniform boundedness can be used to study the convergence of such approximation.

Application 3: approximate quadrature formulae

Example

Suppose that $(q_n)_{n\geq 1}$ is a sequence of quadrature formulae of the form (*), i.e. $q_n(f) = \sum_{j=1}^n w_j^{(n)} f(t_j^{(n)})$. Then

$$\lim_{n\to\infty} q_n(f) = \int_{-1}^1 f(t)\,dt \text{ for all } f \in C[-1,1]$$

if and only if

(i) the convergence holds for monomials:

$$\lim_{n\to\infty} q_n(t^k) = \int_{-1}^1 t^k\,dt \text{ for all } k \geq 0,$$

(ii) and there exists $K \geq 0$ such that

$$\sum_{j=1}^n |w_j^{(n)}| \leq K \text{ for all } n \geq 1.$$

Application 3: approximate quadrature formulae

- ($\Rightarrow$) This appeared in the 2018 exam.
- (i) is clear.
- For (ii), we regard q_n as a linear functional on $X = C[-1,1]$, equipped with the supremum norm. It is routine to check that $q_n \in X^*$ and $\|q_n\|_* \leq \sum_{j=1}^n |w_j^{(n)}|$.
- Testing the norm against a continuous function f with $|f| \leq 1$ such that $f(t_j^{(n)}) = sign(w_j^{(n)})$, we have

$$\|q_n\|_* = \sum_{j=1}^n |w_j^{(n)}|.$$

- So (ii) means that (q_n) is bounded in X^*. But this is a consequence of the principle of uniform boundedness, as $q_n(f)$ is bounded for every f.

Application 3: approximate quadrature formulae

- ($\Leftarrow$) From the discussion made earlier, (ii) means that the sequence (q_n) is bounded in X^*.
- By (i), $q_n(P)$ converges for every polynomial P to the $I(P) := \int_{-1}^{1} P(t)\,dt$. To conclude, we fix $f \in C[-1,1]$ arbitrarity and show that $q_n(f) \to I(f)$.
- We use the fact that the space of polynomials is a dense subspace of $C[-1,1]$.
  - ⋆ For given $\varepsilon > 0$, select a polynomial P such that $\|P - f\| \leq \varepsilon$. This gives $|q_n(f) - q_n(P)| \leq K\varepsilon$ and $|I(f) - I(P)| \leq \|I\|\varepsilon$.
  - ⋆ We next pick N large such that $|q_n(P) - I(P)| \leq \varepsilon$ for all $n \geq N$.
  - ⋆ Then

$$|q_n(f) - I(f)| \leq (1 + K + \|I\|)\varepsilon \text{ for all } n \geq N.$$

This means $q_n(f) \to I(f)$.

Strong convergence of operators

Theorem

Let X and Y be Banach spaces and consider a sequence $T_n \in \mathscr{B}(X, Y)$. The following statements are equivalent.

1. (i) *There exists $T \in \mathscr{B}(X, Y)$ such that, for every $x \in X$, $T_n x \to Tx$ as $n \to \infty$.*
2. (ii) *For each $x \in X$, the sequence $(T_n x)$ is convergent.*
3. (iii) *There is a constant M and a dense subset Z of X such that $\|T_n\| \leq M$ and the sequence $(T_n z)$ is convergent for each $z \in Z$.*

Sketch of proof

- It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Much of the proof of (iii) $\Rightarrow$ (i) is very similar to what we did in the example about approximate quadrature. I'll leave it to you to read the details for in the lecture notes, and discuss the easier statement that (ii) $\Rightarrow$ (i).

Strong convergence of operators

Sketch of proof

- Let $Tx = \lim_{n\to\infty} T_n x$. It is clear that T is linear. So the issue is to show that T is bounded.
- By (ii), the principle of uniform boundedness implies that (T_n) is bounded, say $\|T_n\| \leq K$ for all n.
- In particular $\|T_n x\| \leq K\|x\|$. It follows that

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq K\|x\|$$

and so T is bounded.

Strong convergence of operators

Example

Let (g_n) be a bounded sequence in $L^\infty(\mathbb{R})$ and define $T : L^1(\mathbb{R}) \to \ell^\infty$ by letting $Tf = (x_n)$ where $x_n = \int_{\mathbb{R}} g_n(t)\, f(t)\, dt$. Show that $\mathrm{Im}\, T \subset c_0$ if and only if $\int_a^b g_n \to 0$ for every finite interval (a, b). Give an example in which this holds but it is not the case that $(g_n(t))$ converges a.e.

- This was an exam question in some distant past.
- Let $X = L^1(\mathbb{R})$ and $x_n(f) = \int_{\mathbb{R}} g_n(t)\, f(t)\, dt$. We have $x_n \in X^*$ and $\|x_n\|_* = \|g_n\|_{L^\infty}$, which is bounded.
- $Tf \in c_0$ is equivalent to $x_n(f) \to 0$.
- The first part follows from the theorem and the fact that the span of the set of characteristic functions of open finite interval is dense in $X = L^1(\mathbb{R})$.
- For the last part, consider $g_n(t) = \sin nt$.