B4.2 Functional Analysis II
Lecture 4

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In the last lecture

- The closed linear span of an orthonormal sequence.
- The Riesz representation theorem.
- Adjoint operators.

In this lecture

- Norm of operators on Hilbert spaces.
- Self-ajointness of orthogonal projection operators.
- Isometric and unitary operators.
- Mazur-Ulam's theorem on linearity of isometries.

Norm of operators on Hilbert spaces

Lemma

Let X be a Hilbert space.

1. *If $T \in \mathscr{B}(X)$, then*

$$\|T\|_{\mathscr{B}(X)} = \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}.$$

2. *If $T \in \mathscr{B}(X)$ and T is self-adjoint, then*

$$\|T\|_{\mathscr{B}(X)} = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}.$$

Proof

- By Cauchy-Schwarz' inequality, when $\|x\| = \|y\| = 1$,

$$|\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\| = \|T\|.$$

Norm of operators on Hilbert spaces

Proof

- Let us show that $\|T\| \leq \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}$.
- We will show that $\|T\| - \varepsilon < \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}$ for any small $\varepsilon > 0$.
- Indeed, we first pick $\|x\| = 1$ such that $\|Tx\| > \|T\| - \varepsilon$. Next, we pick y colinear to Tx such that $\|y\| = 1$. Then $|\langle Tx, y\rangle| = \|Tx\|\|y\| = \|Tx\| > \|T\| - \varepsilon$.
- We have thus shown (i):

$$\|T\| = \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}.$$

Norm of operators on Hilbert spaces

Proof

- Let us turn to (ii). Let

$$K = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$$

  and we need to show that $\|T\| = K$.
  We note that, by linearity,

$$|\langle Tx, x\rangle| \leq K\|x\|^2 \text{ for all } x \in X.$$

- By (i), $K \leq \|T\|$. To conclude, we show that $\|T\| - \varepsilon < K$ for any $\varepsilon > 0$.
- By (i), we can select $\|x\| = \|y\| = 1$ such that

$$\|T\| - \varepsilon < |\langle Tx, y\rangle|.$$

- Replacing y by ay for some scalar a with $|a| = 1$, we may assume that $|\langle Tx, y\rangle| = \langle Tx, y\rangle = \mathrm{Re}\langle Tx, y\rangle$.

Norm of operators on Hilbert spaces

Proof

- We now use polarisation to express $\langle Tx, y\rangle$ in terms of expressions of the form $\langle Tu, u\rangle$:

$$4\mathrm{Re}\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle.$$

By the observation made earlier on, the right hand side is dominated by

$$\leq K\|x+y\|^2 + K\|x-y\|^2 = 2K(\|x\|^2 + \|y\|^2) = 4K.$$

- Putting things together, we have

$$\|T\| - \varepsilon \leq |\langle Tx, y\rangle| = \mathrm{Re}\,\langle Tx, y\rangle \leq K,$$

which concludes the proof.

Norm of operators on Hilbert spaces

Proposition

Let X be a Hilbert space and $A \in \mathscr{B}(X)$. Then

$$\|A^* A\|_{\mathscr{B}(X)} = \|A\|^2_{\mathscr{B}(X)}.$$

In particular, if A is self-adjoint, then $\|A^2\|_{\mathscr{B}(X)} = \|A\|^2_{\mathscr{B}(X)}$.

Proof: By part (ii) of the lemma,

$$\begin{aligned} \|A^* A\| &= \sup\{|\langle A^* A x, x\rangle| : \|x\| = 1\} \\ &= \sup\{|\langle Ax, Ax\rangle| : \|x\| = 1\} \\ &= \sup\{\|Ax\|^2 : \|x\| = 1\} = \|A\|^2. \end{aligned}$$

Self-adjointness of orthogonal projection operators

Theorem

Let X be a Hilbert space and Y and Z are its closed subspaces such that $X = Y \oplus Z$. Let $P : X \to Y$ be the induced direct sum projection, i.e. $P(y + z) = y$. Then the following are equivalent.

(i) $Z = Y^\perp$.

(ii) $P^* = P$.

(iii) $\|P\| \leq 1$ *(and in such case* $\|P\| = 1$ *or* $P \equiv 0$*).*

Proof: Sheet 2.

Unitary operators

Definition

A linear operator between two Hilbert spaces is called unitary if it is isometric and surjective.

Note that isometricity implies injectivity.

Proposition

Let X, Y be Hilbert spaces and $T, U \in \mathscr{B}(X, Y)$. Then

$$T \text{ is isometric} \Leftrightarrow \langle Tx, Ty \rangle = \langle x, y \rangle \text{ for all } x, y \in X \Leftrightarrow T^*T = I_X,$$

and

$$\begin{aligned} U \text{ is unitary} &\Leftrightarrow U^*U = I_X \text{ and } UU^* = I_Y \\ &\Leftrightarrow \text{Both } U \text{ and } U^* \text{ are isometric.} \end{aligned}$$

Example of unitary operators

Example:

- $X = \mathbb{C}^n$, $Y = \mathbb{C}^m$ and $A \in \mathscr{B}(X, Y)$ is given by $Ax = Mx$ with $M \in \mathbb{C}^{m \times n}$. Then $A^* y = \bar{M}^t y$. Hence A is isometric if and only if $\bar{M}^t M = I_n$ (which implies $n \leq m$) and A is unitary if and only if $m = n$ and M is a unitary matrix.
- Let $X = \ell^2$ and L and R be the left-shift and right-shift operators on X. We knew that $L^* = R$. As $RL \neq I$ but $LR = I$, we have that R is isometric but not unitary while L is neither isometric nor unitary. (These can also be checked directly.)
- Let $X = L^2(\mathbb{R})$ and $M_h \in \mathscr{B}(X)$ be the multiplication operator $M_h f = hf$ where $h \in L^\infty(\mathbb{R})$. As $M_h^* = M_{\bar{h}}$, M_h is isometric if and only if M_h is unitary if and only if $|h| = 1$ a.e.

Unitary operators on $L^2(-\pi,\pi)$

Theorem (Bochner)

Let U be a unitary operator on $L^2(-\pi,\pi)$. Then there is a function $K : (-\pi,\pi)^2 \to \mathbb{C}$ such that

$$\int_{-\pi}^{s} Uf(t)\,dt = \int_{-\pi}^{\pi} \overline{K(s,t)}\,f(t)\,dt \text{ for all } f \in L^2(-\pi,\pi).$$

Loosely speaking, the theorem says $Uf(s) = \frac{d}{ds}\int_a^b \overline{K(s,t)}\,f(t)\,dt$.

Proof

- For $s \in (-\pi,\pi)$, let $e_s = \chi_{(-\pi,s)}$.
- Then

$$\int_{-\pi}^{s} Uf(t)\,dt = \langle Uf, e_s\rangle = \langle f, U^* e_s\rangle = \int_{-\pi}^{\pi} f(t)\overline{(U^* e_s)(t)}\,dt.$$

The conclusion follows with $K(s,t) = (U^* e_s)(t)$.

Linearity of isometry

Proposition

Let X and Y be (real or complex) Hilbert spaces. If a map $T : X \to Y$ is an isometry and $T(0) = 0$, then T is linear over $\mathbb{R}$.

Remark

When $X = Y = \mathbb{C}^2$, the map $T(z_1, z_2) = (z_1, \bar{z}_2)$ is a surjective isometry satisfying $T(0) = 0$, but T is not linear (nor skew-linear) over $\mathbb{C}$.

Proof

- It suffices to show that $T(\frac{1}{2}(x + y)) = \frac{1}{2}(T(x) + T(y))$ for all $x, y \in X$.
- If $x = y$, the assertion is clear. Suppose henceforth that $x \neq y$.

Linearity of isometry

Proof

- Let $z = \frac{1}{2}(x+y)$. We have
$$\|T(x) - T(y)\| = \|x - y\|,$$
$$\|T(z) - T(x)\| = \|z - x\| = \frac{1}{2}\|y - x\|,$$
$$\|T(z) - T(y)\| = \|z - y\| = \frac{1}{2}\|y - x\|.$$
- Hence $\|T(x) - T(y)\| = \|T(z) - T(x)\| + \|T(z) - T(y)\|$.
- We thus have a situation where the triangle inequality is saturated. Squaring and expanding, this leads to a situation where Cauchy-Schwarz' inequality is saturated:

$$\mathrm{Re}\langle T(x) - T(z), T(z) - T(y)\rangle = \|T(x) - T(z)\|\|T(z) - T(y)\|.$$

- This is possible only if $T(x) - T(z)$ and $T(z) - T(y)$ are linearly dependent.

Linearity of isometry

Proof

- ... $T(x) - T(z)$ and $T(z) - T(y)$ are linearly dependent. Without loss of generality, we assume for some (real or complex) scalar λ that $T(x) - T(z) = \lambda(T(z) - T(y))$.
- As $\|T(x) - T(z)\| = \|T(z) - T(y)\| \neq 0$, we have $|\lambda| = 1$. Also, as $\|T(x) - T(y)\| = \|T(z) - T(x)\| + \|T(z) - T(y)\|$, we have $|\lambda + 1| = 2$.
- We deduce that $\lambda = 1$ and $T(z) = \frac{1}{2}(T(x) + T(y))$ as desired.

Metric characterisation of midpoints

We next give metric characterisation of the midpoint between two points in a normed vector space.

- Suppose x, y are two given points in a normed vector space X and $z = \frac{1}{2}(x + y)$ be their midpoint.
- When X is Hilbert, z is the only point which is half way between x and y, i.e. $\|z - x\| = \|z - y\| = \frac{1}{2}\|x - y\|$.
- In general the set

$$H = \Big\{ w : \|w - x\| = \|w - y\| = \frac{1}{2}\|x - y\| \Big\}$$

  may contain points other than z.
- We note that H is symmetric about z: If $w \in H$, then $2z - w \in H$. This is because $(2z - w) - x = y - w$ and $(2z - w) - y = x - w$.

Metric characterisation of midpoints

- We now construct a nested sequence $H = H_0 \supset H_1 \supset H_2 \supset \dots$ with $diam(H_{n+1}) \leq \frac{1}{2} diam(H_n)$, all of which contains z and are symmetric about z. In particular $\bigcap H_n = \{z\}$.
- As stated we let $H_0 = H \ni z$.
- By the symmetricity of H_0 around z, we have

$$\|z - w\| = \frac{1}{2}\|(2z - w) - w\| \leq \frac{1}{2} diam(H_0) \text{ for all } w \in H_0.$$

- We define the sets H_n's inductively by

$$H_{n+1} = \Big\{p \in H_n : \|p - w\| \leq \frac{1}{2} diam(H_n) \text{ for all } w \in H_n\Big\}.$$

Metric characterisation of midpoints

- $H_{n+1} = \left\{ p \in H_n : \|p - w\| \leq \frac{1}{2} diam(H_n) \text{ for all } w \in H_n \right\}$.
- To finish the construction, we need to show that if H_n contains z and is symmetric about z, then the same conclusions hold for H_{n+1}.
  - ⋆ If $w \in H_n$, then $(2z - w) \in H_n$ and so

    $$\|z - w\| = \frac{1}{2}\|(2z - w) - w\| \leq \frac{1}{2} diam(H_n),$$

    which implies that $z \in H_{n+1}$.
  - ⋆ Suppose $p \in H_{n+1}$ and $w \in H_n$, then as $2z - w \in H_n$,

    $$\|(2z-p)-w\| = \|(2z-w)-p\| = \|p-(2z-w)\| \leq \frac{1}{2} diam(H_n).$$

    So $2z - p \in H_{n+1}$, i.e. H_{n+1} is symmetric about z.

Metric characterisation of midpoints

- To summarise, we have constructed for every two points x, y a nested sequence $H_0 \supset H_1 \supset \dots$ which 'converges' to $\{z\}$, where the definition of the sets H_n's uses only the metric structure of X. The linear structure is used only in the proof of the convergence.

As an application, we have

Theorem (Mazur and Ulam)

Let X and Y be (real or complex) normed vector spaces. If $T : X \to Y$ is a surjective isometry and $T(0) = 0$, then T is linear over $\mathbb{R}$.

Proof: Fix two points x, y in X. As a *surjective* isometry, T maps the sets H_n's of the pair (x, y) to the corresponding sets for the pair $T(x)$ and $T(y)$. By continuity, T preserves the limits of those sets, i.e. $T(\frac{1}{2}(x + y)) = \frac{1}{2}(T(x) + T(y))$. The conclusion follows.

Nonlinear non-surjective isometries

Remark

Note that in Mazur and Ulam's result, we assumed that T is surjective. This was not required in the Hilbert space setting. This assumption cannot be dropped in general.

Example: Let $X = \mathbb{R}$ and $Y = \mathbb{R}^2$ equipped with the norm

$$\|(x_1, x_2)\|_\infty = \max(|x_1|, |x_2|).$$

Let $T(x) = (x, f(x))$ where f is any differentiable functions with $f(0) = 0$ and $|f'(x)| \leq 1$.

- $T(0) = 0$ and T is nonlinear.
- Note that by the mean value theorem, $|f(x) - f(y)| = |f'(c)||x - y| \leq |x - y|$.
- $\|T(x) - T(y)\|_\infty = \max(|x - y|, |f(x) - f(y)|) = |x - y|$, so T is an isometry.