B4.2 Functional Analysis II
Lecture 7

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In the last lecture

- The open mapping theorem.
- The inverse mapping theorem.

In this lecture

- The closed graph theorem.
- The equivalence of the open mapping theorem, the inverse mapping theorem and the closed graph theorem.

The closed graph theorem

Theorem (Closed graph theorem)

Let X and Y be Banach spaces and T be a linear operator from X into Y. Then T is bounded if and only if its graph

$$\Gamma(T) = \{(x, y) \in X \times Y : y = Tx\}$$

is closed in $X \times Y$.

Remark

- Here $X \times Y$ is equipped with the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$.
- It is clear that $\Gamma(T)$ is a subspace of $X \times Y$. An equivalent way to state the theorem is: T is bounded if and only if $\Gamma(T)$ is a Banach subspace of $X \times Y$.

The closed graph theorem

Remark

- Usually, to show that a map $A : X \to Y$ is continuous, one needs to show that if $x_n \to x$, then $A(x_n) \to A(x)$. This breaks down to the convergence of $(A(x_n))$ and the identification of the limit as $A(x)$.
  In the case A is linear, the closed graph theorem says that one can assume the first leg, i.e. the convergence of $(A(x_n))$ in this discussion. Indeed, $\Gamma(A)$ is closed means that if $(x_n, y_n) \in \Gamma(A)$ and $(x_n, y_n) \to (x, y)$, then $(x, y) \in \Gamma(A)$. This means that if $x_n \to x$ and $A(x_n) = y_n \to y$, then $y = A(x)$.

The closed graph theorem

Proof

- ($\Rightarrow$) Suppose T is bounded. If $(x_n, y_n) \in \Gamma(T)$ and $(x_n, y_n) \to (x, y)$, then $x_n \to x$ and $y_n \to y$. But as T is bounded, we also have that $y_n = Tx_n \to Tx$. It follows that $y = Tx$, i.e. $(x, y) \in \Gamma(T)$. This proves that $\Gamma(T)$ is closed.
- ($\Leftarrow$) Conversely, suppose $\Gamma(T)$ is closed in $X \times Y$ so that $\Gamma(T)$ is a Banach subspace of $X \times Y$.
  - ⋆ Define the operators $P_1 : \Gamma(T) \to X$ and $P_2 : \Gamma(T) \to Y$ by
  $$P_1(x, Tx) = x \text{ and } P_2(x, Tx) = Tx.$$
  Clearly these maps are bounded and with norm ≤ 1.
  - ⋆ P_1 is a linear bijection. By the inverse mapping theorem, P_1 has a bounded inverse P_1^{-1}.
  - ⋆ We now have $T = P_2 \circ P_1^{-1}$ and so T is bounded.

Application 1: Continuity of direct sum projection

Example

Let X be a Banach space, and Y and Z are closed subspaces of X such that $X = Y \oplus Z$. Then the direct sum projection $P : X \to Y$ from X onto the first summand Y is bounded.

- This was used in Lecture 2 on a theorem on the existence of complementary spaces.
- By the closed graph theorem, we need to show that if $x_n \to x$ and $Px_n \to y$, then $y = Px$.
- Let $y_n = Px_n \in Y$ and $z_n = x_n - y_n \in Z$.
- Since Y is closed, $y = \lim y_n \in Y$.
- Likewise $x - y = \lim z_n \in Z$.
- As $x = y + (x - y) \in Y + Z$, we have $Px = y$ as desired.

Application 2: Self-adjointness $\Rightarrow$ boundedness

Example

Let X be a Hilbert space and $T : X \to X$ be a linear mapping. If $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in X$, then T is bounded (and so self-adjoint).

- We again use the closed graph theorem: We suppose $x_n \to x$ and $Tx_n \to z$ and would like to show that $z = Tx$.
- For $y \in X$, we have

$$\langle Tx, y\rangle = \langle x, Ty\rangle = \lim_{n\to\infty} \langle x_n, Ty\rangle = \lim_{n\to\infty} \langle Tx_n, y\rangle = \langle z, y\rangle.$$

- We deduce $z = Tx$ as wanted.

Application 3: Multiplication operators

Example

Suppose h is a measurable function on $\mathbb{R}$ such that $M_h f := hf \in L^1(\mathbb{R})$ for all $f \in L^1(\mathbb{R})$. Show that $h \in L^\infty(\mathbb{R})$.

- Let $X = L^1(\mathbb{R})$.
- It is clear that the converse holds: If $h \in L^\infty(\mathbb{R})$, then M_h maps X into itself. In fact $M_h \in \mathscr{B}(X)$.
- Let us first show that if M_h maps X into itself then M_h is bounded on X using the closed graph theorem.
- Suppose $f_n \to f$ and $M_h f_n \to g$, and we aim to show $g = M_h f$.
- Since $f_n \to f$ in L^1, there is subsequence $f_{n_j} \to f$ a.e. It follows that $M_h f_{n_j} \to M_h f$ a.e. Here one is using the fact that the set $\{h = \pm\infty\}$ has zero measure (check this!).

Application 3: Multiplication operators

- Since $M_h f_{n_j} \to g$ in L^1, there is a subsequence $M_h f_{n'_j} \to g$ a.e.
- We deduce that $g = M_h f$ a.e. and so $M_h \in \mathscr{B}(X)$.
- We proceed to show that $h \in L^\infty(\mathbb{R})$.
- We start by writing the fact that M_h is bounded:

$$\int_{\mathbb{R}} |h||f|\, dx \leq \|M_h\| \int_{\mathbb{R}} |f|\, dx \text{ for all } f \in X. \qquad (*)$$

  We claim that $|h| \leq \|M_h\|$ a.e., that is

  the set $Z := \{x : |h(x)| > \|M_h\|\}$ has zero measure.

- We add some buffer room by considering a slightly smaller set

$$Z_\varepsilon := \{x : |h(x)| > \|M_h\| + \varepsilon\} \text{ with } \varepsilon > 0.$$

  Clearly if Z has zero measure, so does Z_ε. Conversely, if Z_ε has zero measure for all ε, then Z also has zero measure by the monotone convergence theorem.

Application 3: Multiplication operators

- We have

$$\int_{\mathbb{R}} |h||f|\,dx \le \|M_h\| \int_{\mathbb{R}} |f|\,dx \text{ for all } f \in X. \qquad (*)$$

- We'd like to show that $|Z_\varepsilon| = |\{x : |h(x)| > \|M_h\| + \varepsilon\}| = 0$.
- If we knew that Z_ε has finite measure, we could use $f = \chi_{Z_\varepsilon}$ in (*) to obtain

$$(\|M_h\| + \varepsilon)|Z_\varepsilon| \le \int_{Z_\varepsilon} |h|\,dx \le \|M_h\||Z_\varepsilon|,$$

where we have used $|h| > \|M_h\| + \varepsilon$ in Z_ε.
This implies $|Z_\varepsilon| = 0$.

Application 3: Multiplication operators

- We'd like to show that $|Z_\varepsilon| = |\{x : |h(x)| > \|M_h\| + \varepsilon\}| = 0$.
- Since we do not know if Z_ε has finite measure, we have to truncate.
  Fix some $n > 0$ and let $Z_{\varepsilon,n} = Z_\varepsilon \cap [-n, n]$. Using $f = \chi_{Z_{\varepsilon,n}}$ in (*) as above, we have

  $$(\|M_h\| + \varepsilon)|Z_{\varepsilon,n}| \leq \|M_h\||Z_{\varepsilon,n}|$$

  Again, we obtain $|Z_\varepsilon \cap [-n, n]| = 0$. The monotone convergence theorem then implies that $|Z_\varepsilon| = 0$.
- We conclude that $h \in L^\infty(\mathbb{R})$ and $|h| \leq \|M_h\|$ a.e.

Application 4: Integral operators

Example

Let $K\colon [0,1]\times[0,1] \to \mathbb{R}$ be measurable. Assume that

($\star$) there is a set of measure zero $Z \subset [0,1]$ such that, for each $x \in [0,1] \setminus Z$ and $f \in L^2(0,1)$, the function $K(x,\cdot)\,f\colon y \mapsto K(x,y)f(y)$ is integrable, and

($\star\star$) for each $f \in L^2(0,1)$, the function Tf defined by

$$(Tf)(x) = \int_0^1 K(x,y)f(y)\,dy, \qquad x \in (0,1),$$

belongs to $L^2(0,1)$.

Show that $f \mapsto Tf$ defines a bounded linear operator from $L^2(0,1)$ into itself.

- This was in the 2018 exam.

Application 4: Integral operators

Sketch

- The problem requires 2 applications of the closed graph theorem.
- In the first leg, one uses $(\star)$ to show that, for every $x \in Z^c$, $f \mapsto K(x,\cdot)f$ is bounded linear operator from $L^2(0,1)$ into $L^1(0,1)$.
- In the second leg one uses the above and $(\star\star)$ to show that T is bounded: If $f_n \to f$ in L^2 and $Tf_n \to h$ in L^2, then
  - ▷ On one hand $K(x,\cdot)f_n \to K(x,\cdot)f$ in L^1 for every $x \in Z^c$, which implies $Tf_n \to Tf$ in Z^c,
  - ▷ On the other hand, we also have that a subsequence Tf_{n_j} converges to h a.e.
  - ▷ As $|Z| = 0$, we have $h = Tf$ a.e.
- We conclude that T is bounded.

Counterexample

Example

Let $X = C^1[0,1]$ and $Y = C[0,1]$ and equipped both with the supremum norm so that X is incomplete and Y is complete. Let $D : X \to Y$ be the differentiation map: $Du = u'$. Show that D is linear, has closed graph but is not bounded.

- It is clear that D is linear. Also, if $u_n(x) = x^n$, then $\|u_n'\|_Y = n$ while $\|u_n\|_X = 1$. So D is unbounded.
- We show that $\Gamma(D)$ is closed: If $u_n \to u$ in X and $Du_n \to v$ in Y, then $v = Du$.
- A caveat is that we are not given the convergence of u_n to u in C^1, though both u_n and u are C^1. In easy term, we need to show that if $(u_n) \subset C^1[0,1]$ is such that u_n converges uniformly to $u \in C^1[0,1]$ and u_n' converges uniformly to $v \in C[0,1]$, then $u' = v$.

Counterexample

- ... we need to show that if $(u_n) \subset C^1[0,1]$ is such that u_n converges uniformly to $u \in C^1[0,1]$ and u_n' converges uniformly to $v \in C[0,1]$, then $u' = v$.
- We have
$$u_n(x) = u_n(0) + \int_0^x u_n'(t)\,dt.$$
- Then using the uniform convergence of u_n and u_n', we can send $n \to \infty$ to obtain
$$u(x) = u(0) + \int_0^x v(t)\,dt.$$
- The fundamental theorem of calculus then gives $u' = v$ as desired.

Almost projection

Theorem

Let X be a Banach space and Y and Z be closed subspaces so that $Y + Z$ is also closed. Then there exists a constant $C \geq 0$ such that for every $w \in Y + Z$ can be decomposed as $w = y + z$ with

$$y \in Y, \quad z \in Z, \quad \|y\| \leq C\|w\| \quad \textit{and} \quad \|z\| \leq C\|w\|.$$

Proof

- Equip $Y \times Z$ with the product norm $\|(y, z)\| = \|y\| + \|z\|$ so that $Y \times Z$ is a Banach space.
- Consider the map $S : Y \times Z \to Y + Z$ defined by

$$S(y, z) = y + z.$$

S is linear, surjective and bounded (by triangle inequality).

Almost projection

Proof

- By the open mapping theorem $S(B_{Y\times Z}(0,1))$ contains a ball $B_{Y+Z}(0,r)$.
- This means that every element $w \in Y+Z$ with $\|w\| < r$ can be written as $w = y+z$ with $\|y\| + \|z\| < 1$.
- In particular, every element $w \in Y+Z$ with $\|w\| \leq r/2$ can be written as $w = y+z$ with $\|y\| + \|z\| < 1$.
- Rescaling gives that every element $w \in Y+Z$ can be written as $w = y+z$ with $\|y\| + \|z\| \leq \frac{2}{r}\|w\|$. This proves the theorem with $C = \frac{2}{r}$.

CGT $\Leftrightarrow$ IMT $\Leftrightarrow$ OMT

Remark

In our treatment as well as in many texts, one proves first the open mapping theorem and use it to deduce the inverse mapping theorem and then the closed graph theorem. It turns out that they are all equivalent.

- (CGT $\Rightarrow$ IMT) Suppose that the closed graph theorem hold and let $T \in \mathscr{B}(X, Y)$ be a bijection. Since T is continuous its graph $\Gamma(T)$ is closed. As $\Gamma(T^{-1})$ can be obtained from the graph of T by swapping to the coordinates:

$$\Gamma(T^{-1}) = \{(y, x) \in Y \times X : (x, y) \in \Gamma(T)\},$$

$\Gamma(T^{-1})$ is closed. By the closed graph theorem, T^{-1} is bounded.

CGT $\Leftrightarrow$ IMT $\Leftrightarrow$ OMT

- (IMT $\Rightarrow$ OMT) Suppose that the inverse mapping theorem hold and let $T \in \mathscr{B}(X, Y)$ be a surjection. We want to show that T is open.
  - ⋆ Let $N = \text{Ker}\, T$ and $\hat{X} = X/N$ be the quotient space. Note that the quotient map $\pi : X \to \hat{X}$ is bounded, linear, surjective and open. (Check this!)
  - ⋆ T descends to a bounded bijection $\hat{T} \in \mathscr{B}(\hat{X}, Y)$ giving the commutative diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & Y \arrow[dl, dashed, bend left, "\hat{T}^{-1}"] \\ \hat{X} \arrow[ur, bend left, "\hat{T}"] & \end{tikzcd}$$

  - ⋆ Now if $U \subset X$ is open, then $T(U)$ is the preimage of the open set $\pi(U)$ under the continuous map T^{-1} and so is open.