B4.2 Functional Analysis II
Lecture 3

Luc Nguyen
luc.nguyen@maths

University of Oxford

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In the last lecture

- The projection theorem
- Pythagoras' theorem
- Bessel's inequality

In this lecture

- The closed linear span of an orthonormal sequence.
- The Riesz representation theorem.
- Adjoint operators.

The closed linear span of an orthonormal sequence

Theorem

Let $S = \{x_1, x_2, \ldots\}$ be an infinite orthonormal sequence in an infinite-dimensional Hilbert space X. Then

$$\overline{Span(S)} = \left\{ x = \sum_{n=1}^{\infty} a_n\, x_n \middle| (a_n) \in \ell^2 \right\}$$

where the sum $\sum_{n=1}^{\infty} a_n\, x_n$ converges in the sense of the Hilbert space norm. Furthermore $a_n = \langle x, x_n \rangle$ and

$$\|x\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2. \qquad \textit{(Parserval's identity)}$$

The closed linear span of an orthonormal sequence

Proof

- Let $Y = \overline{Span(S)}$ and

$$Z = \left\{ x = \sum_{n=1}^{\infty} a_n\, x_n \middle| (a_n) \in \ell^2 \right\}.$$

- Note that if $(a_n) \in \ell^2$, then, by Pythagoras' theorem,

$$\Big\| \sum_{n=N_1}^{N_2} a_n\, x_n \Big\|^2 = \sum_{n=N_1}^{N_2} |a_n|^2 \stackrel{N_1, N_2 \to \infty}{\longrightarrow} 0.$$

Hence, the sequence $\left(\sum_{n=1}^{N} a_n\, x_n \right)_N$ is Cauchy and so the sum $\sum_{n=1}^{\infty} a_n\, x_n$ is well defined. In particular $Z \subset Y$.

The closed linear span of an orthonormal sequence

Proof

- Conversely, assume that $x \in Y$ and we'll show that $x \in Z$.
- Let $a_n = \langle x, x_n \rangle$. By Bessel's inequality, $(a_n) \in \ell^2$ and so the vector $\tilde{x} := \sum_{n=1}^\infty a_n\, x_n \in Z \subset Y \subset X$.
- Now $\langle \tilde{x}, x_m \rangle = \lim\limits_{N\to\infty} \langle \sum\limits_{n=1}^N a_n\, x_n, x_m \rangle = a_m = \langle x, x_m \rangle$.

  Thus $x - \tilde{x}$ is perpendicular to all x_n. This means that $x - \tilde{x} \in Span(S)^\perp = \overline{Span(S)}^\perp = Y^\perp$.
- It follows that $x - \tilde{x} \in Y \cap Y^\perp = \{0\}$ and so $x = \tilde{x} \in Z$.
- Finally, by Pythagoras' theorem,

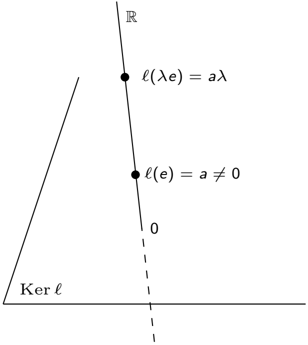
$$\|x\|^2 = \sum_{n=1}^N |\langle x, x_n \rangle|^2 + \Big\| x - \sum_{n=1}^N \langle x, x_n \rangle x_n \Big\|^2 \xrightarrow{N\to\infty} \sum_{n=1}^\infty |\langle x, x_n \rangle|^2,$$

  which proves Parseval's identity.

The Riesz representation theorem

Motivation:

- Let $X = \mathbb{R}^n$ and consider $\ell \in X^*$.
- If $\ell \neq 0$, then the kernel of ℓ has codimension one: $X = \mathrm{Ker}\,\ell \oplus \mathbb{R}$.
- It is easy to see that the ℓ restricted to the $\mathbb{R}$-summand is simply a multiplication of the scalar component along this summand by a nontrivial scalar constant.



- If we equip $\mathbb{R}^n$ with its standard inner product and arrange so that the splitting $X = \mathrm{Ker}\,\ell \oplus \mathbb{R}$ is orthogonal and e has unit length on the $\mathbb{R}$-summand, then the above implies that

$$\ell(y) = \langle y, ae \rangle.$$

The Riesz representation theorem

Theorem (Riesz representation theorem)

Let X be a (real or complex) Hilbert space and ℓ be a bounded linear functional on X. Then ℓ is of the form

$$\ell(y) = \langle y, x\rangle \text{ for all } y \in X$$

for some $x \in X$. Furthermore, the point x is uniquely determined and $\|x\| = \|\ell\|_*$.

Remark

It follows that X and X^* are isometrically isomorphic in the real case and isometrically anti-isomorphic in the complex case via the map

$$x \in X \mapsto \ell_x \in X^*, \qquad \ell_x(y) = \langle y, x\rangle.$$

The Riesz representation theorem

Proof

- The proof is very much the same as the proof we saw in the finite dimensional case.
- If $\ell = 0$, then $x = 0$. Assume henceforth that $\ell \neq 0$.
- Let Y be the kernel of ℓ. Then Y is a closed subspace of X.
- By the projection theorem, $X = Y \oplus Y^\perp$.
- Since $Y^{\perp\perp} = Y$ is a strict subspace of X (as $\ell \not\equiv 0$), $Y^\perp$ contains a non-zero element, say e, which we may assume to have unit length.
- Note that $a := \ell(e) \neq 0$.
- Then for any $y \in X$, we have $y - \frac{1}{a}\ell(y)e \in \mathrm{Ker}\,\ell = Y$.
- Taking inner product with e yields $\langle y, e\rangle - \frac{1}{a}\ell(y) = 0$. This gives the representation with $x = \bar{a}e$.

The Riesz representation theorem

Proof

- The uniqueness is clear: If $\ell(y) = \langle y, x\rangle = \langle y, \tilde{x}\rangle$ for all y, then $\langle y, x - \tilde{x}\rangle = 0$ for all y and so $x - \tilde{x} = 0$.
- Finally, we show that $\|\ell\|_* = \|x\|$. On one hand, we have by Cauchy-Schwarz' inequality that

$$\ell(y) = \langle y, x\rangle \leq \|y\|\|x\|$$

and so $\|\ell\|_* \leq \|x\|$.
On the other hand, we have

$$\|x\|^2 = \langle x, x\rangle = \ell(x) \leq \|\ell\|_*\|x\|$$

and so $\|x\| \leq \|\ell\|_*$.
We deduce that $\|\ell\|_* = \|x\|$.

Adjoint operators

Definition

Let X and Y be Hilbert spaces and $A \in \mathscr{B}(X, Y)$. An adjoint operator of A, denoted A^*, is an operator belonging to $\mathscr{B}(Y, X)$ such that

$$\langle Ax, y\rangle_Y = \langle x, A^*y\rangle_X.$$

Proposition

Every operator $A \in \mathscr{B}(X, Y)$ has a unique adjoint operator $A^* \in \mathscr{B}(Y, X)$.

Definition

An operator $A \in \mathscr{B}(X)$ is said to be *self-adjoint* if $A^* = A$.

Adjoint operators

Proof of the proposition:

- For every fixed $y \in Y$, the map $x \mapsto \langle Ax, y\rangle_Y =: T_y(x)$ belongs to X^* with norm $\|T_y\|_{X^*} \leq \|A\|\|y\|_Y$.
- By the Riesz representation theorem, there is a unique $A^*y \in X$ such that $\langle Ax, y\rangle_Y = T_y(x) = \langle x, A^*y\rangle_X$ with $\|A^*y\|_X = \|T_y\|_{X^*} \leq \|A\|\|y\|_Y$.
- The linearity of A^* is clear. We thus have $A^* \in \mathscr{B}(Y, X)$.

Adjoint operators

Remark

The Riesz representation theorem gives two isometric (anti-)isomorphism $\psi_X : X^* \to X$ and $\psi_Y : Y^* \to Y$. The adjoint operator A^* is related to the dual operator A' by the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "A^*"] & X \\ Y^* \arrow[u, "\psi_Y"] \arrow[r, "A'"] & X^* \arrow[u, "\psi_X"'] \end{tikzcd}$$

(Recall that $A' : Y^* \to X^*$ is defined by $(A'\ell)(x) = \ell(Ax)$ for all $x \in X$ and $\ell \in Y^*$.)

Properties of adjoint operators

Proposition

We list some properties of the adjoint operators:

1. (i) $\|A\|_{\mathscr{B}(X,Y)} = \|A^*\|_{\mathscr{B}(Y,X)}$.
2. (ii) $A^{**} = A$.
3. (iii) *If* $A, B \in \mathscr{B}(X, Y)$ *and* $a, b \in \mathbb{C}$, *then* $(aA + bB)^* = \bar{a}\, A^* + \bar{b}\, B^*$.
4. (iv) *If* $T \in \mathscr{B}(X, Y)$ *and* $S \in \mathscr{B}(Y, Z)$, *then* $(ST)^* = T^* S^*$.
5. (v) $I_X^* = I_X$.
6. (vi) *If* $A \in \mathscr{B}(X, Y)$, *then* $Ker\, A = (Im\, A^*)^\perp$ *and* $\overline{Im\,(A)} = Ker\, A^*$.
7. (vii) $A \in \mathscr{B}(X)$ *is invertible with a bounded inverse if and only if* A^* *is invertible with a bounded inverse.*

Proof of (vii)

- Suppose that $A^* \in \mathscr{B}(X)$ is invertible with a bounded inverse. We would like to show that A is invertible.
- Once this is done, we can apply this same statement to A^* to obtain the converse.
- Let $B = ((A^*)^{-1})^*$. We have

$$\langle ABx, y\rangle = \langle Bx, A^*y\rangle = \langle x, B^*A^*y\rangle = \langle x, (A^*)^{-1}A^*y\rangle = \langle x, y\rangle.$$

Since this is true for all y, we have that $ABx = x$ for all x and so A is invertible with inverse $A^{-1} = B$.

Examples of adjoint operators

Example: Consider a linear operator $A : \mathbb{C}^n \to \mathbb{C}^m$ given in the standard bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ by a matrix $M \in \mathbb{C}^{m\times n}$, i.e. $Ax = Mx$ for $x \in \mathbb{C}^n$.

- The adjoint operator: We have

$$\langle Ax, y\rangle = (Mx)^t \bar{y} = x^t M^t \bar{y} = x^t \overline{\bar{M}^t y} = \langle x, A^* y\rangle.$$

  This means $A^* y = \bar{M}^t y$, and so the matrix corresponding to A^* is the conjugate transpose of the matrix corresponding to A.

- The dual operator: If $(\mathbb{C}^n)^*$ and $(\mathbb{C}^m)^*$ has dual bases $e_1^*, \ldots, e_n^*$ and $f_1^*, \ldots, f_m^*$, then

$$\Big(A'(\sum y_\ell f_\ell^*)\Big)(\sum x_k e_k) = \sum y_\ell f_\ell^*(\sum M_{ik} x_k f_i) = \sum y_i M_{ik} x_k.$$

  This means $A' y^* = M^t y^*$ and so the the matrix corresponding to A' is the transpose of the matrix corresponding to A.

Example of adjoint operators

Example: Let $X = Y = L^2(0,1)$ and A be the integral operator

$$(Af)(x) = \int_0^1 k(x,y)f(y)\,dy$$

where $k : (0,1)^2 \to \mathbb{C}$ is a given bounded measurable function.

- Then A is a linear operator of $L^2(0,1)$ into itself. In fact, for every $f \in L^2(0,1)$, one has $Af \in L^\infty(0,1)$ as

  $$|Af(x)| \leq \|k\|_{L^\infty} \int_0^1 |f(y)|\,dy \leq \|k\|_{L^\infty} \|f\|_X.$$

  This implies that

  $$\|Af\|_X \leq \|Af\|_{L^\infty} \leq \|k\|_{L^\infty} \|f\|_X$$

  and so $A \in \mathscr{B}(X)$.

Example of adjoint operators

- Let us compute the adjoint of A. We write

$$\langle Af, g\rangle = \int_0^1 \Big\{ \int_0^1 k(x,y)\, f(y)\, dy \Big\} \bar{g}(x)\, dx.$$

- We woud like to apply Fubini's theorem. To do so, we need to check that $k(x,y)\, f(y)\, \bar{g}(x) \in L^1((0,1)^2)$. As k is bounded, we only need to check that $|f(y)|\, |g(x)|$ has finite integral over $(0,1)^2$. This is a consequence of Tonelli's theorem:

$$\begin{aligned}
\int_{(0,1)^2} |f(y)|\, |g(x)|\, dx\, dy &= \int_0^1 \Big\{ \int_0^1 |f(y)|\, |g(x)|\, dy \Big\} dx \\
&= \int_0^1 \Big\{ |g(x)| \int_0^1 |f(y)|\, dy \Big\} dx \\
&\leq \int_0^1 \Big\{ |g(x)| \|f\|_X \Big\} dx \leq \|f\|_X \|g\|_X.
\end{aligned}$$

Example of adjoint operators

- We can now continue our computation using Fubini's theorem:

$$\begin{aligned}\langle Af, g\rangle &= \int_0^1 \Big\{ \int_0^1 k(x,y)\, f(y)\, dy \Big\} \bar{g}(x)\, dx \\ &= \int_0^1 f(y) \Big\{ \int_0^1 k(x,y) \bar{g}(x)\, dx \Big\}\, dy \\ &= \int_0^1 f(y) \overline{\int_0^1 \overline{k(x,y)}\, g(x)\, dx}\, dy = \langle f, A^* g\rangle.\end{aligned}$$

- We conclude that

$$(A^* g)(x) = \int_0^1 \overline{k(y,x)} g(y)\, dy.$$

- In particular, A is self-adjoint if and only if $k(x,y) = \overline{k(y,x)}$.

Example of adjoint operators

Example: Let $X = \ell^2$ and L and R be the left-shift and right-shift operators. Then $L = R^*$ and $R = L^*$.

Example: Let $X = L^2(\mathbb{R}; \mathbb{C})$ and $h : \mathbb{R} \to \mathbb{C}$ be a bounded measurable function. Let M_h denote the multiplication operator:

$$M_h f(x) = h(x) f(x).$$

Then $M_h \in \mathscr{B}(X)$ and

$$M_h^* = M_{\bar{h}}.$$