B4.2 Functional Analysis II
Lecture 13

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In the last lecture

- Divergence of (some) Fourier series in $C_{per}(\mathbb{R})$.
- Completeness of the trigonometric system in $C_{per}(\mathbb{R})$.
- Completeness of the trigonometric system in $L^p(-\pi, \pi)$ for $1 \leq p < \infty$.
- Convergence of Fourier series in $L^p(-\pi, \pi)$ for $1 < p < \infty$.

In this lecture

- Condition for convergence of Fourier series at a point for functions in $L^1(-\pi, \pi)$.
- Cesaro convergence of partial Fourier sums in $C_{per}(\mathbb{R})$.

Hölder continuity

- Suppose f is defined in an open interval I containing a point x_0. For a given $\alpha \in (0,1]$, we say that f is α-Hölder continuous at x_0 if there exist $A > 0$ and $\delta_0 > 0$ such that

$$|f(x_0+h) - f(x_0)| \leq A|h|^\alpha \text{ for } |h| \leq \delta_0.$$

When $\alpha = 1$, we say f is Lipschitz continuous at x_0.

- When f is only defined almost everywhere, we amend the above definition to: f is α-Hölder continuous at x_0 if there exist $A > 0$, $\delta_0 > 0$ and f_0 such that

$$|f(x_0+h) - f_0| \leq A|h|^\alpha \text{ for a.e. } |h| \leq \delta_0.$$

In such case, it's convenient to redefined $f(x_0)$ to f_0. Note that

$$f_0 = \lim_{h \to 0^+} \frac{1}{2h} \int_{x_0-h}^{x_0+h} f(x)\,dx.$$

Hölder continuity and convergence of Fourier series

Theorem (Dirichlet)

Assume that $f \in L^1(-\pi,\pi)$, f is 2π-periodic and f is α-Hölder continuous at a point x_0 for some $\alpha \in (0,1]$. Then

$$\lim_{N\to\infty} S_N f(x_0) = f(x_0).$$

Remark

With a little bit more effort (check this!), one can adapt the theorem to a situation where f is "left and right" α-Hölder continuous at a point x_0, where one has

$$\lim_{N\to\infty} S_N f(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Hölder continuity and convergence of Fourier series

Proof

- We may take $x_0 = 0$. Since the assertion is linear in f and clearly holds for constant functions, we may also assume that $f(0) = 0$. We thus have to show that $S_N f(0) \to 0$.
- Recall that
  $$S_N f(x) = \int_{-\pi}^{\pi} f(t)\, k_N(x-t)\, dt \text{ where } k_N(x) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$
  In particular, since k_N is even,
  $$\begin{aligned} S_N f(0) &= \int_{-\pi}^{\pi} f(t)\, k_N(-t)\, dt \\ &= \int_{-\pi}^{0} f(t)\, k_N(t)\, dt + \int_{0}^{\pi} f(t)\, k_N(t)\, dt \\ &= \int_{0}^{\pi} (f(t) + f(-t))\, k_N(t)\, dt. \end{aligned}$$

Hölder continuity and convergence of Fourier series

Proof

- $S_N f(0) = \int_0^\pi (f(t) + f(-t))\, k_N(t)\, dt.$
- Heuristic: Observe the singular behavior of $k_N(t)$ near $t = 0$:

$$k_N(t) = \frac{1}{2\pi} \frac{\sin(N + \frac{1}{2})t}{\sin \frac{t}{2}} \sim \frac{\text{oscillatory in } [-1, 1] \text{ for large } N}{t}.$$

  - ⋆ If f is merely continuous, the integral above is morally $\int_0^\pi \frac{o(1)}{t}\, dt$ which is difficult to bound, and in fact resulting in the divergence result we knew.
  - ⋆ If f is α-Hölder continuous, we are lead to $\int_0^\pi \frac{O(1)}{t^{1-\alpha}}\, dt$ which is bounded uniformly in N.

- The proof proceeds by refining the above idea using ‘divide and conquer’ technique.

Hölder continuity and convergence of Fourier series

Proof

- $S_N f(0) = \displaystyle\int_0^\pi (f(t) + f(-t))\, k_N(t)\, dt.$
- Fix some small δ for the moment. For $t \in (0, \delta)$, we use the inequality $\sin \frac{t}{2} \geq \frac{t}{\pi}$ to estimate

$$|k_N(t)| \leq \frac{1}{2\pi} \frac{1}{\sin \frac{t}{2}} \leq \frac{1}{2t}.$$

The α-Hölder continuity of f at 0 gives $|f(t) + f(-t)| \leq 2At^\alpha$ in $(0, \delta)$ provided $\delta < \delta_0$ which we will assume.
Therefore

$$\Big| \int_0^\delta (f(t) + f(-t))\, k_N(t)\, dt \Big| \leq \int_0^\delta At^{\alpha - 1} = A\alpha^{-1}\delta^\alpha.$$

Hölder continuity and convergence of Fourier series

Proof

- $S_N f(0) = \displaystyle\int_0^\pi (f(t) + f(-t))\, k_N(t)\, dt.$
- $\Big| \displaystyle\int_0^\delta (f(t) + f(-t))\, k_N(t)\, dt \Big| \leq A\alpha^{-1}\delta^\alpha.$
- It remains to consider $J_{N,\delta} := \displaystyle\int_\delta^\pi (f(t) + f(-t))\, k_N(t)\, dt.$

  We write

  $$k_N(t) = \frac{1}{\sin\frac{t}{2}} \sin(Nt + \frac{t}{2}) = \cos Nt + \cot\frac{t}{2} \sin Nt.$$

  Hence,

  $$J_{N,\delta} = \int_{-\pi}^{\pi} [g_\delta(t) \cos Nt + h_\delta(t) \sin Nt]\, dt$$

  where
  $$g_\delta(t) = \chi_{(\delta,\pi)}(t)(f(t) + f(-t)),$$
  $$h_\delta(t) = \chi_{(\delta,\pi)}(t)(f(t) + f(-t)) \cot\frac{t}{2}.$$

Hölder continuity and convergence of Fourier series

Proof

- $S_N f(0) = \int_0^\pi (f(t) + f(-t))\, k_N(t)\, dt$.
- $\Big| \int_0^\delta (f(t) + f(-t))\, k_N(t)\, dt \Big| \leq A\alpha^{-1}\delta^\alpha$.
- $J_{N,\delta} = \int_{-\pi}^\pi [g_\delta(t) \cos Nt + h_\delta(t) \sin Nt]\, dt$

  where
  $$g_\delta(t) = \chi_{(\delta,\pi)}(t)(f(t) + f(-t)),$$
  $$h_\delta(t) = \chi_{(\delta,\pi)}(t)(f(t) + f(-t)) \cot \frac{t}{2}.$$

- For fixed $\delta > 0$, $g_\delta, h_\delta \in L^1(-\pi,\pi)$.
  By Riemann-Lebesgue's lemma, we therefore have $J_{N,\delta} \to 0$ as $N \to \infty$.

Hölder continuity and convergence of Fourier series

Proof

- $S_N f(0) = \int_0^\pi (f(t) + f(-t))\, k_N(t)\, dt$.
- $\Big| \int_0^\delta (f(t) + f(-t))\, k_N(t)\, dt \Big| \leq A\alpha^{-1}\delta^\alpha$.
- For fixed $\delta > 0$, $\int_\delta^\pi (f(t) + f(-t))\, k_N(t)\, dt \to 0$ as $N \to \infty$.
- We thus have that, for every $\delta > 0$,

$$\limsup_{N\to\infty} |S_N f(0)| \leq A\alpha^{-1}\delta^\alpha.$$

Sending $\delta \to 0$, we conclude that $S_N f(0) \to 0$ as desired.

Cesaro convergence of partial Fourier sums

Although the partial Fourier sums of a continuous function does not necessarily converge uniformly, we have the following result:

Theorem (Féjer)

For every $f \in C_{per}(\mathbb{R})$ it holds that

$$\sigma_N f := \frac{S_0 f + \ldots + S_N f}{N+1} \to f \text{ in } C_{per}(\mathbb{R}) \text{ as } N \to \infty.$$

Ideas of proof

- We write the partial Fourier sums as a convolution $S_N f = k_N * f$. It follows that $\sigma_N f = F_N * f$ where

$$F_N(x) = \frac{1}{N+1}(k_0(x) + \ldots + k_N(x)) = \frac{1}{2\pi(N+1)} \frac{1 - \cos(N+1)x}{1 - \cos x}.$$

Cesaro convergence of partial Fourier sums

Ideas of proof

- The F_N's are called Féjer kernels. They behave better than Dirichlet kernels in a number of way:
  - ⋆ $F_N \geq 0$.
  - ⋆ $\|F_N\|_{L^1(-\pi,\pi)} = 1$.
  - ⋆ For $\delta < x < \pi$, $0 \leq F_N(x) \leq \frac{1}{\pi(N+1)(1-\cos\delta)}$.
- Using the above properties, one can follow the same proof as in the last to reach the conclusion. Details are left as an exercise.

Examples

Example

Let $f \in L^\infty(-\pi, \pi)$ and (c_n) its Fourier coefficients. For $p < q \in \mathbb{Z}$, define a bilinear form $A : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{C}$ by

$$A(x, y) = \sum_{m,n=p}^{q} c_{n+m} x_n y_m$$

Show that $|A(x, y)| \leq \|f\|_{L^\infty} \|x\| \|y\|$. Deduce Hilbert's inequality

$$\Big| \sum_{m,n=0}^{N} \frac{x_n y_m}{m + n + 1} \Big| \leq \pi \|x\| \|y\| \text{ for all } N \geq 0.$$

Examples

- By polarisation, it suffices to bound $|A(x,x)|$. The trick is to recognise $A(x,x)$ as $\int_{-\pi}^{\pi} f(t)(P(t))^2\,dt$ where $P(t) = \sum_{n=p}^{q} x_n e^{-inx}$.
- Then $|A(x,x)| \leq \|f\|_{L^\infty}\|P\|_{L^2}^2 \leq \|f\|_{L^\infty}\|x\|^2$ where we have use Pythagoras' theorem for the last inequality.
- To obtain Hilbert's inequality, we need to select $f \in L^\infty$ such that $c_n = \frac{1}{n+1}$ for $n \geq 0$.
  - ⋆ If we attempt to sum $\sum_{n\geq 0} \frac{1}{n+1} e^{inx}$, we have an issue with boundedness at $x = 0$:

$$\sum_{n=0}^{\infty} \frac{1}{n+1} e^{inx} = e^{-ix} \sum_{m=1}^{\infty} \frac{1}{m} e^{imx} = ie^{-ix} \int \sum_{m=1}^{\infty} e^{imx}\,dx$$

$$= ie^{-ix} \int \frac{e^{ix}}{1-e^{ix}}\,dx = -e^{-ix}\text{"}\ln(1-e^{ix})\text{"},$$

where the integration constant should be chosen appropriately.

Examples

- … we need to select $f \in L^\infty$ such that $c_n = \frac{1}{n+1}$ for $n \geq 0$.
  - ⋆ We fix the issue by adding in $c_n = \frac{1}{n+1}$ for $n \leq -2$ too:

$$
\begin{aligned}
f(x) &= \sum_{n \neq -1}^{\infty} \frac{1}{n+1} e^{inx} = e^{-ix} \sum_{m=1}^{\infty} \frac{1}{m}(e^{imx} - e^{-imx}) \\
&= 2ie^{-ix} \mathrm{Im} \sum_{m=1}^{\infty} \frac{1}{m} e^{imx} = 2ie^{-ix} \text{``}\mathrm{Im} \ln(1 - e^{ix})\text{''} \\
&= ie^{-ix} \text{``}(\pi - x + 2\pi\mathbb{Z})\text{''}.
\end{aligned}
$$

  - ⋆ The branch cut (i.e. integration constant) is chosen so that the zeroth Fourier coefficients of $f(x)e^{ix}$ is zero. This leads to

$$
f(x) = \begin{cases} ie^{-ix}(\pi - x) & \text{if } 0 < x < \pi, \\ ie^{-ix}(-\pi - x) & \text{if } -\pi < x < 0. \end{cases}
$$

Examples

Example

There exists a function $f \in L^1(-\pi,\pi)$ whose Fourier series is

$$f \sim \sum_{j=0}^{\infty} \left(e^{i\sqrt{j+1}} - e^{i\sqrt{j}}\right) \cos(j!t).$$

Prove that for every $t \in \pi\mathbb{Q}$, the sequence $(S_N(t))$ diverges, but the sequence $(\sigma_N f(t))$ converges. Is f bounded and continuous? Does f lie in $L^2(-\pi,\pi)$?
[*You may assume that* $\frac{1}{N+1}\sum_{n=0}^{N} e^{i\sqrt{n+1}} \to 0$ *as* $N \to \infty$.]

- This was an exam question in some distant past.
- If $\frac{t}{\pi}$ is rational, then there is some large N_0 such that $j!\frac{t}{\pi}$ is an even integer for all $j \geq N_0$.

Examples

$$f \sim \sum_{j=0}^{\infty} \left(e^{i\sqrt{j+1}} - e^{i\sqrt{j}}\right) \cos(j!t).$$

- If $\frac{t}{\pi}$ is rational, then ... $j!t$ is an even integer for all $j \geq N_0$.
- It follows that

$$S_N f(t) - S_{N_0-1} f(t) = \sum_{j=N_0}^{N} \left(e^{i\sqrt{j+1}} - e^{i\sqrt{j}}\right) = e^{i\sqrt{N+1}} - e^{i\sqrt{N_0}}.$$

- It follows that $(S_N f(t))$ diverges, since e.g. $e^{i\sqrt{N+1}}$ can be close to 1 and -1 infinitely frequently (check this!).
- The convergence of $(\sigma_N f(t))$ also follows:

$$\sigma_N f(t) = \frac{N_0}{N+1} \sigma_{N_0-1} f(t) + \frac{N - N_0 + 1}{N+1} (S_{N_0-1} f(t) - e^{i\sqrt{N_0}})$$
$$+ \frac{1}{N+1} \sum_{j=N_0}^{N} e^{i\sqrt{N+1}} \to S_{N_0-1} f(t) - e^{i\sqrt{N_0}}.$$

Examples

$$f \sim \sum_{j=0}^{\infty} \left(e^{i\sqrt{j+1}} - e^{i\sqrt{j}}\right) \cos(j!t).$$

- For the last bit, we show that $f \notin L^2(-\pi,\pi)$ (and hence is not bounded nor continuous).
- Indeed, if $f \in L^2(-\pi,\pi)$, we would have by Parseval's identity that

$$A := \sum_{j=0}^{\infty} |b_{j!}|^2 < \infty \text{ where } b_{j!} = e^{i\sqrt{j+1}} - e^{i\sqrt{j}}.$$

- Now

$$b_{j!} = e^{i\sqrt{j}}(e^{i(\sqrt{j+1}-\sqrt{j})} - 1) = e^{i\sqrt{j}}(e^{\frac{i}{\sqrt{j+1}+\sqrt{j}}} - 1).$$

It follows that $|b_{j!}| \sim j^{-1/2}$ for large j, and so A is in fact infinite. We conclude that $f \notin L^2(-\pi,\pi)$.