B4.2 Functional Analysis II
Lecture 16

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In the last 2 lectures

- The ABCs of spectral theory for bounded linear operators.
- Spectra of normal operators.
- Spectra of self-adjoint operators.
- Spectra of unitary operators.

In this lecture

- More on spectra of normal operators.
- A big example: spectra of integral operators.

Normal operators: Recap

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be normal.
We knew

- $\sigma_r(T) = \emptyset$ and $\sigma(T) = \sigma_{ap}(T) = \sigma_p(T) \cup \sigma_c(T)$,
- If x and y are eigenvectors of T corresponding to different eigenvalues, then $\langle x, y \rangle = 0$.
- $\mathrm{rad}(\sigma(T)) = \|T\|$.

Isolated spectral points of normal operators

Proposition

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be normal. If $\lambda \in \sigma(T)$ is an isolated point of the spectrum of T, then $\lambda \in \sigma_p(T)$.

Ideas of proof

- Without loss of generality, we suppose that $\lambda = 0$.
- We will use the following fact: When T is normal and $\lambda \notin \sigma(T)$, $\|R_\lambda(T)\| = \frac{1}{dist(\lambda, \sigma(T))}$. (See Sheet 4.)
- Hence, if we define $S_\lambda := \lambda R_\lambda(T)$ for $\lambda \in \rho(T)$, then $\|S_\lambda\| = 1$ for all small λ and S_λ is analytic in $\lambda \in \rho(T)$.
- FACT: It can be proved from the above that, as $\lambda \to 0$, $S_\lambda \to S$ in $\mathscr{B}(X)$ in norm.

Isolated spectral points of normal operators

Ideas of proof

- ... $S_\lambda := \lambda R_\lambda(T)$ ...
- FACT: as $\lambda \to 0$, $S_\lambda \to S$ in $\mathscr{B}(X)$.
- Now, observe that, for $x \in X$,

$$\begin{aligned} \lambda x - \lambda S x + TSx &= \lambda(\lambda I - T)R_\lambda(T)x - (\lambda I - T)Sx \\ &= (\lambda I - T)(S_\lambda x - Sx). \end{aligned}$$

- Sending $\lambda \to 0$, we get $TSx = 0$ for all x.
- This implies that $\mathrm{Ker}\, T \supset \mathrm{Im}\, S \neq 0$ (since $\|S\| = \lim \|S_\lambda\| = 1$). We conclude that $\lambda = 0$ is an eigenvalue of T.

Other results for normal operators

Let X be a complex Hilbert space and $T \in \mathscr{B}(X)$ be normal.

- If $\sigma(T) = \{\lambda_1, \lambda_2, \ldots\}$ is countable, then every $x \in H$ has a unique expansion of the form
$$x = \sum_{i=1}^{\infty} x_i$$
where $Tx_i = \lambda_i x_i$ and $\langle x_i, x_j \rangle$ if $i \neq j$.
- (Nieminen) T is self-adjoint if and only if $\sigma(T) \subset \mathbb{R}$.
- (Donaghue) T is unitary if and only if $\sigma(T) \subset \{|\lambda| = 1\}$.

Other results for normal operators

Theorem (Fuglede)

Suppose X is a Hilbert space and $S, T \in \mathscr{B}(X)$ are normal. If S and T commute, then $S + T$ and ST are normal.

Idea of proof

- It boils down to prove that S and T^* commute.
- Let us consider the simplest case where $\sigma(T)$ is finite. In this case, we know that each $\lambda \in \sigma(T)$ is an eigenvalue of T and hence X decomposes into an orthogonal direct sum of eigenspaces $X = X_1 \oplus \ldots \oplus X_N$. The proof proceeds then as in finite dimensional setting.
- The key is to observe that that S preserves X_i: If X_i is the eigenspace corresponding to an eigenvalue λ and $x \in X_i$, then $\lambda Sx = STx = TSx$ and so $Sx \in X_i$. The commutativity of S and T^* follows.

Other results for normal operators

Idea of proof

- In the general case, the proof along the above line is possible but difficult.
- It is in fact more convenient to find a new route. (It is instructive to think of S and T still as square matrices of finite size in the argument to follow.)
  We start with the observation that $ST^n = T^nS$ for all $n \geq 0$. This leads to
  $$S\exp(i\bar{\lambda}T) = \exp(i\bar{\lambda}T)S \text{ for all } \lambda \in \mathbb{C}.$$
  where the exponential function is defined using power series. (The reason why we choose to put $\bar{\lambda}$ instead of λ will be clear in a moment.)
- Supposing that we know $\exp(i\bar{\lambda}T)^{-1} = \exp(-i\bar{\lambda}T)$, this gives
  $$S = \exp(-i\bar{\lambda}T)S\exp(i\bar{\lambda}T).$$

Other results for normal operators

Idea of proof

- $S = \exp(-i\bar{\lambda}T)S\exp(i\bar{\lambda}T)$.
  This then gives

$$\exp(-i\lambda T^*)S\exp(i\lambda T^*)$$
$$= \exp(-i\lambda T^*)\exp(-i\bar{\lambda}T)S\exp(i\bar{\lambda}T)\exp(i\lambda T^*).$$

- As T is normal, $\exp(-i\lambda T^*)\exp(-i\bar{\lambda}T) = \exp(-i(\lambda T^* + \bar{\lambda}T))$ and so the above becomes

$$\exp(-i\lambda T^*)S\exp(i\lambda T^*)$$
$$= \exp(-i(\lambda T^* + \bar{\lambda}T))S\exp(i(\lambda T^* + \bar{\lambda}T)).$$

Other results for normal operators

Idea of proof

- ... $\exp(-i\lambda T^*) S \exp(i\lambda T^*)$
$$= \exp(-i(\lambda T^* + \bar{\lambda} T)) S \exp(i(\lambda T^* + \bar{\lambda} T)).$$
- Now note that $\lambda T^* + \bar{\lambda} T$ is self-adjoint. It is fairly easy to check from this that $\exp(\pm i(\lambda T^* + \bar{\lambda} T))$ is unitary. Hence the right hand side of the equation above is bounded.
- We thus have that $F(\lambda) := \exp(-i\lambda T^*) S \exp(i\lambda T^*)$ is bounded for all $\lambda \in \mathbb{C}$. Since F is analytic, a suitable Liouville theorem then tells us that F is constant, i.e. $F(\lambda) = S$ for all λ.
- The assertion that $ST^* = T^*S$ follows by taking derivative in λ and setting $\lambda = 0$.

Example

Example

Let $X = L^2(0,1)$ and suppose k is a uniformly continuous function on $[0,1]^2$. Let $T \in \mathscr{B}(X)$ be given by

$$Tf(x) = \int_0^1 k(x,y)f(y)\,dy.$$

Discuss condition for T to be normal or self-adjoint, and discuss the spectral properties of T.

- The adjoint operator is

$$T^*f(x) = \int_0^1 \overline{k(z,x)}f(z)\,dz.$$

Thus T is self-adjoint if $\overline{k(z,x)} = k(x,z)$ for all x, z.

Example

- We have

$$T^*Tf(x) = \int_0^1 \int_0^1 \overline{k(z,x)}k(z,y)f(y)\,dz\,dy,$$
$$TT^*f(x) = \int_0^1 \int_0^1 \overline{k(y,z)}k(x,z)f(y)\,dz\,dy.$$

  Thus T is normal if $\int_0^1 \overline{k(z,x)}k(z,y)\,dz = \int_0^1 \overline{k(y,z)}k(x,z)\,dz$ for all x, y.

- T has the following important property:

  Every bounded sequence $(f_n) \subset X$ has a subsequence (f_{n_k}) such that (Tf_{n_k}) converges in X. (*)

Example

- We need

Theorem (Kolmogorov-Riesz-Fréchet)

A bounded sequence $(g_n) \in L^2(\mathbb{R})$ has a convergent subsequence if for every $\varepsilon > 0$, there exists δ such that $\|g_n(\cdot + h) - g_n\|_{L^2(\mathbb{R})} \leq \varepsilon$ for every n and every h with $|h| \leq \delta$.

- If suffices to consider $h > 0$.
  When $x \in (0, 1 - h)$, we have

$$|Tf_n(x+h) - Tf_n(x)| \leq \int_0^1 |k(x+h, y) - k(x, y)||f_n(y)|\, dy$$

By uniform continuity, this can be made smaller than $o(1)\|f_n\|$ by squeezing h.

Example

- When $x \in (0, 1-h)$, $|Tf_n(x+h) - Tf_n(x)| \leq o(1)\|f_n\|$.
- When $x \in (-h, 0)$,

$$|Tf_n(x+h) - Tf_n(x)| \leq \int_0^1 |k(x+h,y)||f_n(y)|\,dy \leq \|k\|_{L^\infty}\|f_n\|.$$

- Likewise, when $x \in (1-h, 1)$,

$$|Tf_n(x+h) - Tf_n(x)| \leq \int_0^1 |k(x,y)||f_n(y)|\,dy \leq \|k\|_{L^\infty}\|f_n\|.$$

- Altogether, we have

$$\|Tf_n(\cdot + h) - Tf_n\|_{L^2(\mathbb{R})} \leq o(1)\|f_n\| + O(h)\|k\|_{L^\infty}\|f_n\| \leq \varepsilon$$

when $|h| < \delta$ is sufficiently small.

Example

- We have thus verified the needed condition to apply Kolmogorov-Riesz-Fréchet theorem to conclude that

  Every bounded sequence $(f_n) \subset X$ has a subsequence (f_{n_k}) such that (Tf_{n_k}) converges in X. (*)

- With (*) at hand, we can discuss $\sigma_{ap}(T)$:
  - ⋆ If $\lambda \in \sigma_{ap}(T)$, then there exists $\|f_n\| = 1$ such that $\lambda f_n - Tf_n \to 0$.
  - ⋆ From (*), we may assume $Tf_n \to g$. Then $\lambda f_n \to g$. If we have that $\lambda \neq 0$, then $f_n \to \lambda^{-1} g =: f$ and so $\lambda f = Tf$, i.e. $\lambda \in \sigma_p(T)$.
  - ⋆ So $\sigma_{ap}(T)$ is made up of eigenvalues except possibly 0. In particular, $\sigma_c(T) \subset \{0\}$.

Example

- ... we can discuss $\sigma_{ap}(T)$:
  - ⋆ If $(\lambda_n) \subset \sigma_p(T)$ and $\lambda_n \to \lambda$, then we can apply the above argument to a sequence of unimodular eigenvectors (f_n) of T corresponding to λ_n (since $\lambda f_n - Tf_n = (\lambda - \lambda_n) f_n \to 0$). We have the dichotomy: either f_n converges or $\lambda = 0$.
  - ⋆ If T is normal,[1] eigenvectors corresponding to different eigenvalues are orthogonal, (f_n) is not Cauchy. Hence $\lambda = 0$.
  - ⋆ In the general case, we don't know if the eigenvectors are orthogonal. We amend using Riesz's lemma (from FA1): We select f_{n+1} such that $dist(f_{n+1}, Span(\{f_1, \ldots, f_n\})) \geq \frac{1}{2}$. We again obtain $\lambda = 0$.
  - ⋆ We conclude that $\sigma_p(T)$ is at most countable with 0 being the only possible accumulation point.
  - ⋆ The above argument also shows that if $\lambda \in \sigma_p(T) \setminus \{0\}$, then $\mathrm{Ker}\,(\lambda I - T)$ is finite dimensional.

[1] In the lecture, there is an oversight error here. See the next point.

Example

- Next, we consider $\sigma_r(T)$:
  - ⋆ If T is normal, $\sigma_r(T)$ is empty.
  - ⋆ In the general case, we have $\sigma_r(T) \subset \sigma'_p(T^*)$. Since T^* is also an integral operator of the same form, we have that $\sigma_r(T)$ is also at most countable with 0 being the only possible accumulation point, and, for each $\lambda \in \sigma_r(T) \setminus \{0\}$, the space $\mathrm{Im}(\lambda I - T)$ has finite co-dimension.
- In summary:
  - ⋆ The spectrum of T is either finite, or is of the form $(\lambda_n)_{n=1}^\infty \cup \{0\}$ where λ_n is a sequence converging to zero.
  - ⋆ If $0 \neq \lambda \in \sigma_p(T)$, then $\dim \mathrm{Ker}\,(\lambda I - T)$ is finite.
  - ⋆ If $0 \neq \lambda \in \sigma_r(T)$, then $\mathrm{codim}\,\mathrm{Im}\,(\lambda I - T)$ is finite.
  - ⋆ $\sigma_c(T) \subset \{0\}$.
  - ⋆ If T is normal, $\sigma_r(T) = \emptyset$.
  - ⋆ Note that, by a remark in Lecture 14, the above implies that $\mathrm{Im}\,(\lambda I - T) \neq X$ for all $\lambda \in \sigma(T)$.