B4.2 Functional Analysis II
Lecture 8

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In the last 3 lectures

- The Baire category theorem.
- The principle of uniform boundedness.
- The open mapping theorem.
- The inverse function theorem.
- The closed graph theorem.

In this lecture

- Weak convergence: Examples and basic properties.

Motivation

- One of the most important results we learnt in first year analysis is the following:

Theorem (Bolzano-Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

- Equivalently, this can be rephrased as: Every closed bounded subset of $\mathbb{R}^n$ is sequentially compact. (Note that the closely related result of Heine-Borel gives compactness instead of sequential compactness. These two notions coincide in a metric space.)
- An important application is:

Theorem

A continuous function on a closed bounded subset of $\mathbb{R}^n$ attains its maximum and minimum values.

Motivation

- What about infinite dimension? Unfortunately, the statement is generally wrong: a closed bounded set in an infinite dimensional normed vector space need not be sequentially compact.
- For example, let X be an infinite-dimensional Hilbert space and (x_n) be an orthonormal sequence. Then

$$\|x_n - x_m\| = \sqrt{2} \text{ for all } n \neq m,$$

  and so no subsequence of (x_n) is Cauchy!
- More generally, we learnt in FAI that closed bounded sets in a normed vector space X are compact if and only if $\dim X < \infty$.
- Moral: Perhaps a less stringent notion of convergence is desirable!

Weak convergence

Definition

Let X be a normed vector space.
A sequence $(x_n) \subset X$ is said to converges weakly to $x \in X$ if

$$\lim_{n\to\infty} \ell(x_n) = \ell(x) \text{ for all } \ell \in X^*.$$

We use a half arrow to indicate this convergence: $x_n \rightharpoonup x$.

Sometimes, to stress the difference, we refer to the convergence in norm as strong convergence:

- $x_n \to x$: (x_n) converges (strongly) to x.
- $x_n \rightharpoonup x$: (x_n) converges weakly to x.

Weak convergence

Example

- If $x_n \to x$, then $x_n \rightharpoonup x$. This is because if $\ell \in X^*$, then

$$|\ell(x_n) - \ell(x)| = |\ell(x_n - x)| \leq \|\ell\|_* \|x_n - x\| \to 0.$$

- If X is finite dimensional, then $x_n \to x$ if and only if $x_n \rightharpoonup x$. This can be seen by testing the definition of weak convergence using the functionals which gives the components of x.
- If (x_n) is an orthonormal sequence in a Hilbert space X. Then $x_n \rightharpoonup 0$ and (x_n) does not converge strongly.
  - ⋆ (x_n) doesn't converge strongly because it's not Cauchy: $\|x_n - x_m\| = \sqrt{2}$ if $n \neq m$.
  - ⋆ Pick any bounded linear functional $\ell \in X^*$. We want to show that $\ell(x_n) \to 0$.

Weak convergence

Example

- If (x_n) is an orthonormal sequence in a Hilbert space X. Then $x_n \rightharpoonup 0$ and (x_n) does not converge strongly.
  - ⋆ By the Riesz representation theorem, we can express $\ell(x) = \langle x, y \rangle$ for $x \in X$.
  - ⋆ So we need to show $\langle x_n, y \rangle \to 0$. But this is a consequence of Bessel's inequality:

$$\sum_{n=1}^{\infty} |\langle x_n, y \rangle|^2 \leq \|y\|^2.$$

Weak convergence in ℓ^1

Example (Schur property)

If a sequence (x_n) converges weakly in ℓ^1, then it converges strongly.

Sketch

- This is part of Sheet 3, where you'll also consider weak convergence in ℓ^p for other p's.
- Let us consider the statement that $x_n \rightharpoonup 0$ implies $x_n \to 0$.
- If strong convergence doesn't hold, then, up to extracting a subsequence, we have $\|x_n\| > \varepsilon > 0$ for some ε.
- In the sheet, there is a hint to construct increasing sequences (n_k) and (m_k) such that $\sum_{j \leq m_{k-1}} |x_{n_k}(j)| < \varepsilon/8$ and $\sum_{j \geq m_k} |x_{n_k}(j)| < \varepsilon/8$.
- Once this is done, we test the weak convergence against the element $b \in \ell^\infty = (\ell^1)^*$ given by $b(j) = 1$ for $j \leq m_1$ and $b(j) = sign(x_{n_k}(j))$ if $m_{k-1} < j \leq m_k$.

Weak convergence in ℓ^1

Sketch

- ... $\sum_{j \le m_{k-1}} |x_{n_k}(j)| < \varepsilon/8$ and $\sum_{j \ge m_k} |x_{n_k}(j)| < \varepsilon/8$.
- ... $b(j) = sign(x_{n_k}(j))$ if $m_{k-1} < j \le m_k$.
- We have, for large $k \ge 2$,

$$0 \approx b(x_{n_k}) = \sum_{j \le m_{k-1}} b(j) x_{n_k}(j) + \sum_{m_{k-1} < j \le m_k} |x_{n_k}(j)| + \sum_{j > m_k} b(j) x_{n_k}(j).$$

and

$$\varepsilon < \|x_{n_k}\| = \sum_{j \le m_{k-1}} |x_{n_k}(j)| + \sum_{m_{k-1} < j \le m_k} |x_{n_k}(j)| + \sum_{j > m_k} |x_{n_k}(j)|.$$

- Note that each of the blue term has modulus $\le \varepsilon/8$. We deduce that $b(x_{n_k}) \ge \|x_{n_k}\| - \varepsilon/2 > \varepsilon/2$, which gives a contradiction.

Weak vs. strong convergence

- In Sheet 3, you'll show that if X is Hilbert space, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.
- In the same sheet, you'll show that this remains true if X is a uniformly convex Banach space. Examples of such spaces are ℓ^p and L^p spaces with $1 < p < \infty$.
- We have seen that in ℓ^1, weak and strong convergences are equivalent.
- The discussion in L^1 is more complicated.

Modes of convergence in L^p

Suppose E has finite measure, $1 \leq p < \infty$ and $(f_n) \subset L^p(E)$.

- Strong convergence
  - implies weak convergence,
  - implies convergence in measure,
  - implies almost everywhere convergence after extraction of a subsequence.
  - implies convergence of norms $\|f_n\| \to \|f\|$.
- Almost everywhere convergence
  - implies convergence in measure,
  - implies lower estimate of the liminf of the norms (Fatou's lemma),
  - and convergence of norms imply strong convergence (will be proven in Lecture 10).
- Convergence in measure
  - implies almost everywhere convergence after extraction of a subsequence.

Modes of convergence in L^p

Suppose E has finite measure, $1 \leq p < \infty$ and $(f_n) \subset L^p(E)$.

- Weak convergence
  - ⋆ implies the boundedness of the norms (later in this lecture),
  - ⋆ implies lower estimate of the liminf of the norms (later in this lecture),
  - ⋆ does not necessarily lead to pointwise convergence,
  - ⋆ and convergence of norms in the case $1 < p < \infty$ imply strong convergence (Sheet 3).

A.E. limit = Weak limit in L^p

Example

Suppose $E \subset \mathbb{R}^n$ is measurable, $1 \leq p < \infty$, and $(f_k) \subset L^p(E)$. Show that if $f_k \rightharpoonup f$ in $L^p(E)$ and $f_k \to \tilde{f}$ a.e. in E, then $f = \tilde{f}$ a.e. in E.

- Recall that, for $1 \leq p < \infty$, $(L^p(E))^* \cong L^{p'}(E)$ where p' is the Hölder conjugate exponent of p.
- The statement $f_k \rightharpoonup f$ thus means

$$\int_E f_k\, g \to \int_E f\, g \text{ for all } g \in L^{p'}(E).$$

(We are supposing our functions are real-valued. A conjugation on g is needed in the complex case.)

A.E. limit = Weak limit in L^p

- $\displaystyle\int_E f_k\, g \to \int_E f\, g$ for all $g \in L^{p'}(E)$.
- Suppose that $f_k \not\to f$ a.e. Then the set $\{\tilde f \neq f\}$ has positive measure.
- One of the sets $\{\tilde f > f\}$ and $\{\tilde f < f\}$ must have positive measure. Without loss of generality assume the former.
- By the monotone convergence theorem, there is some $\varepsilon > 0$ such that $A := \{\tilde f > f + \varepsilon\}$ has positive measure.

A.E. limit = Weak limit in L^p

- $\displaystyle\int_E f_k\, g \to \int_E f\, g$ for all $g \in L^{p'}(E)$.
- Suppose not, then ... $A := \{\tilde{f} > f + \varepsilon\}$ has positive measure.
- Recall that $f_k \to \tilde{f}$ a.e. in A. By Egorov's theorem, we can find a set $B \subset A$ of positive measure such that $f_k \to f$ uniformly in B. We can also assume that B has finite measure.
  Therefore, for large k, $|f_k - \tilde{f}| < \varepsilon/2$ and so $f_k > f + \varepsilon/2$ in B.
- Now, choose $g = \chi_B$. We are led to

$$0 = \lim_{k\to\infty} \int_B (f_k - f) > \varepsilon|B|/2 > 0,$$

which is absurd.

Weak convergence $\Rightarrow$ Boundedness

Theorem

A weakly convergent sequence (x_n) in a normed vector space X is uniformly bounded in the norm.

Proof

- This is an application of the principe of uniform boundedness.
- Consider x_n as a linear functional on X^*:

$$T_n(\ell) = \ell(x_n) \text{ for all } \ell \in X^*$$

  so that $\|T_n\|_{**} = \|x_n\|$.
- The space X^* is always complete.
- By the principle of uniform boundedness, the result follows if T_n is pointwise bounded, but this is true as for every $\ell \in X^*$, $(T_n(\ell)) = (\ell(x_n))$ is convergent.

Weak lower semi-continuity of norm

Theorem

Let (x_n) be a sequence in a normed vector space X which converges weakly to some $x \in X$. Then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Proof

- By a result in B4.1 (which is a consequence of the Hahn-Banach theorem), there is some $\ell \in X^*$ such that

$$\|x\| = \ell(x) \text{ and } \|\ell\|_* = 1.$$

- The conclusion follows from the inequality

$$|\ell(x_n)| \leq \|\ell\|_* \|x_n\| = \|x_n\|$$

and the fact that $\ell(x_n) \to \ell(x) = \|x\|$.

Some major results on weak convergence

We will be learning in subsequent lectures the following results:

Theorem (Mazur)

Let K be a closed convex subset of a normed vector space X, (x_n) be a sequence of points in K converging weakly to x. Then $x \in K$.

Theorem (Weak sequential compactness in reflexive Banach spaces)

Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

Theorem (Eberlein)

The closed unit ball in a Banach space X is weakly sequentially compact only if X is reflexive.