8. More ordinal arithmetic

In this handout, I will assume the Axiom of Choice. In particular, I will assume that a countable union of countable sets is countable.

8.1. Failures of continuity on the left

According to the definitions of the operations of ordinal arithmetic, the values of $\alpha+\beta$, $\alpha.\beta$ and α^β increase smoothly as β increases. However they do not increase smoothly as α increases. Instead they move by a series of jerks. For instance, if $n < \omega$ and (for the sake of non-triviality) $n \geq 2$, then $n+\omega$, $n.\omega$ and n^ω all take the same value, namely ω. However, when we reach ω itself, the value of the sum jumps suddenly to $\omega+\omega = \omega.2$, the value of the product lurches upward to $\omega.\omega = \omega^2$, and the value of the exponential jumps even further to $\omega^\omega = \sup\bigl(\omega, \omega^2, \omega^3, \ldots\bigr)$. (Ordinal exponentiation is not defined in the lectures; finding a suitable definition is left as a problem on the problem sheets.)

In fact, ω exhibits the worst kind of asymmetry in the operations of ordinal arithmetic. The operator $\alpha \mapsto \omega + \alpha$ is very well-behaved, being one-to-one, strictly increasing, well-behaved at limits, etc. By contrast, $\alpha + \omega$ is ω itself if $\alpha < \omega$; and then of course $\omega + \omega$ is different.

And ω is not alone. One can prove that β has the same property—that is, for all $\alpha < \beta$, $\alpha + \beta = \beta$—if and only if $\beta = \omega^\delta$ for some δ. And ordinals of the form ω^δ are not hard to come by.

8.2. Cantor Normal Form

The asymmetry of ordinal arithmetic is further exemplified by a rather nice standard form for expressing ordinals derived by Cantor himself. The theorem is that for any ordinal α, there exists a unique finite sequence $\delta_0 > \delta_1 > \cdots > \delta_{k-1}$, and a corresponding sequence n_0, n_1, ..., n_{k-1} of non-zero natural numbers such that

$$\alpha = \omega^{\delta_0} n_0 + \omega^{\delta_1} n_1 + \cdots + \omega^{\delta_{k-1}} n_{k-1}.$$

It would be tempting to say that the powers of ω form a basis. Tempting but, of course, wrong, since ordinal addition is not commutative, so the ordinals are definitely not a vector space. (And the natural numbers are not a field either.)

Cantor published explicit rules for adding and multiplying expressions of this form. Perhaps I can leave them as exercises for the reader. As an example,

$$(\omega^\omega.2 + \omega^5.12 + \omega^3.4 + \omega^2 + \omega.70) + (\omega^3.5 + \omega.8 + 1.11) = \omega^\omega.2 + \omega^5.12 + \omega^3.9 + \omega.8 + 1.11.$$

Notice how the terms ω^2 and $\omega.70$ have been absorbed by the following term in ω^3.

8.3. Fixed points in ordinal arithmetic

We have already seen how ordinals of the form ω^δ are fixed points for addition on the right, in the sense that if $\alpha < \omega^\delta$, then $\alpha + \omega^\delta = \omega^\delta$.

What would the corresponding result be for multiplication?

Suppose that $\beta > \omega$ has the property that for all non-zero $\alpha < \beta$, $\alpha.\beta = \beta$.

For a start, $\beta = \omega.\beta$; it follows that β is a limit ordinal (since, looking at the Cantor Normal Form, an ordinal is a limit if and only if it is of the form $\omega.\zeta$ for some ζ).

Writing $\beta = \omega.\zeta$, we obtain $\beta = \omega.\beta = \omega.\omega.\zeta$; so β is a multiple on the right of ω^2.

In fact we can go further than this, and say that β must be a multiple of ω^ω. And in fact ω^ω is an ordinal of the sort we are looking for. In general, $\beta = \alpha.\beta$ for all non-zero $\alpha < \beta$ if and only if β can be expressed in the form ω^{ω^δ} for some δ.

Let us now try to find fixed points for exponentiation.

Here we run into a problem. It is not that these ordinals do not exist: they do. The problem is how to write them down.

For, suppose $\epsilon > \omega$ has the property that for all α such that $1 < \alpha < \epsilon$, $\epsilon = \alpha^\epsilon$.

Then in particular, $\epsilon = \omega^\epsilon$; so

$$\epsilon = \omega^\epsilon = \omega^{\omega^\epsilon} = \omega^{\omega^{\omega^\epsilon}} = \cdots$$

So how can we write it down? In particular, what is the smallest such ordinal?

The answer is that the methods we have for generating new ordinals by addition, multiplication and exponentiation, have run out of steam. We need new notation. Accordingly, ordinals ϵ having the property that $\epsilon = \omega^\epsilon$ are known as ϵ *numbers*, and the first one, which is the supremum of the sequence $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ is known as ϵ_0. It is possible to prove that an ordinal β is an ϵ number if and only if for all $\alpha < \beta$ such that $\alpha > 1$, $\alpha^\beta = \beta$. The next ϵ-number after ϵ_0 is known as ϵ_1, the next after that as ϵ_2, and so on.

8.4. The problem of naming ordinals

This raises a problem: how can we describe a given ordinal α? Can we find a way to describe it in terms of smaller ordinals by using, for instance, addition, multiplication, exponentiation, the operation $\gamma \mapsto \epsilon_\gamma$, and so on?

The answer, in general, is no. Every principle of nomenclature of this sort that we can come up with will fail at some point, and will indeed fail at a countable ordinal. On the assumption that the human race will only ever do countably much set theory, there will thus be a countable ordinal that will never get a name in the history of the universe.

Let us see why.

One thing that the operators we have used to name large ordinals, such as $\alpha \mapsto \omega^\alpha$ and $\alpha \mapsto \epsilon_\alpha$, have in common, is that they have a property known (for some reason) as *normality*: $\alpha \mapsto \theta_\alpha$ is normal if and only if

1. For all α, $\theta_\alpha < \theta_{\alpha^+}$, and
2. For all limits λ, $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.

Given any such operator, it is not too hard to prove that $\alpha \leq \theta_\alpha$ for all α. Now let α be any ordinal, and consider the sequence

$$\alpha,\ \theta_\alpha,\ \theta_{\theta_\alpha},\ \theta_{\theta_{\theta_\alpha}}, \ldots$$

By the Axiom of Replacement, there is a set containing these ordinals as its elements, and that set then has a supremum λ. And by normality of the operator, $\lambda = \theta_\lambda$.

Thus, in particular, there exists an ordinal λ such that $\epsilon = \epsilon_\lambda$. So, obviously, there does not exist an ordinal $\alpha < \lambda$ such that $\epsilon = \epsilon_\alpha$; this ordinal cannot be described using the operator $\alpha \mapsto \epsilon_\alpha$ and ordinals smaller than itself.

Worse, if we make the additional assumption that whenever α is countable, θ_α is countable also, then we obtain a fixed point $\lambda = \theta_\lambda$ which is countable.

So, we can obtain monsters like the first ordinal λ such that $\lambda = \epsilon_\lambda$, *and this is still a countable ordinal*.

Set theorists have evolved various ways of naming countable ordinals, some of which refer explicitly to how difficult they are to describe. But it is still the case that immediately below the first uncountable ordinal ω_1, there is an interval about which we know relatively little. Indeed, it's not hard to come up with questions about countable ordinals which the Axioms of Set Theory are not powerful enough to answer one way or the other.

8.5. Further reading

There's a great deal more about ordinal arithmetic in Wacław Sierpiński's book *Cardinal and Ordinal Numbers* (PWN, 1958).