PART A TOPOLOGY
HT 2019
EXERCISE SHEET 3

For tutorials in week 7

Compactness

Exercise 1. Let d be one of the metrics d_1, d_2, d_∞ on $\mathbb{R}^n$.

Let K be a non-empty compact subset of $\mathbb{R}^n$, and let F be a non-empty closed subset of $\mathbb{R}^n$. We define

$$\mathrm{dist}(K,F) = \inf\{\mathrm{d}(x,y) : x \in K,\, y \in F\}.$$

Prove that there exists $a \in K$ and $b \in F$ such that $\mathrm{d}(a,b) = \mathrm{dist}(K,F)$.

Exercise 2. Let $(X, \mathcal{T})$ be a topological space and let $C = X \cup \{\infty\}$ where ∞ denotes some extra point not in X. Let $\mathcal{T}'$ denote the union of $\mathcal{T}$ with all subsets of C of the form $V \cup \{\infty\}$ where $V \subseteq X$ and $X \setminus V$ is compact and closed in X.

Prove that $(C, \mathcal{T}')$ is a compact topological space containing $(X, \mathcal{T})$ as a subspace.

This is called the *one-point (or the Alexandrov) compactification* of X.

Prove that the one-point compactification of $\mathbb{R}^2$ is homeomorphic to the 2-sphere.

Exercise 3. Let l^∞ be the space of bounded sequences of real numbers, endowed with the norm $\|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, where $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$.

Prove that the closed unit ball of l^∞, $B'(\mathbf{0},1) = \{\mathbf{x} \in l^\infty \,;\, \|\mathbf{x}\|_\infty \leq 1\}$, is not compact.

Thus, $B'(\mathbf{0},1)$ *is a closed bounded subset in a complete normed vector space which is not compact.*

[*Hint:* The equivalence between compactness and sequential compactness in the setting of metric spaces may be assumed.]

Quotient spaces

Exercise 4. Recall that the *integer part* (or *integral part*) of a real number x is the unique integer $n \in \mathbb{Z}$ such that $n \leq x < n+1$. We denote it by $I(x)$.

On $\mathbb{R}$ we define the relation $x\mathcal{R}y \Leftrightarrow I(x) = I(y)$.

(a) Prove that $\mathcal{R}$ is an equivalence relation.

(b) Let $p : \mathbb{R} \to \mathbb{R}/\mathcal{R}$ be the quotient map, let $\mathbb{R}/\mathcal{R}$ be endowed with the quotient topology, and let U be an open set in $\mathbb{R}/\mathcal{R}$. Prove that if $n \in \mathbb{Z}$ is such that $p(n) \in U$ then $p(n-1) \in U$.

(c) Deduce that the open sets in $\mathbb{R}/\mathcal{R}$ are $\emptyset$, $\mathbb{R}/\mathcal{R}$ and the image sets $p(-\infty, n]$, where $n \in \mathbb{Z}$.

(d) Consider the map $I : \mathbb{R} \to \mathbb{Z}$, $x \mapsto I(x)$. Is the map I continuous (when $\mathbb{Z}$ is endowed with the subspace topology) ?

Prove that I defines a bijection $\widetilde{I} : \mathbb{R}/\mathcal{R} \to \mathbb{Z}$. What is the topology on $\mathbb{Z}$ making $\widetilde{I}$ a homeomorphism ?

Exercise 5. (1) Let X be a topological space and A a subset of X. On $X \times \{0,1\}$ define the partition composed of the pairs $\{(a,0),(a,1)\}$ for $a \in A$, and of the singletons $\{(x,i)\}$ if $x \in X \setminus A$ and $i \in \{0,1\}$.

Let $\mathcal{R}$ be the equivalence relation defined by this partition, let Y be the quotient space $[X \times \{0,1\}]/\mathcal{R}$ and let $p : X \times \{0,1\} \to Y$ be the quotient map.

(a) Prove that there exists a continuous map $f : Y \to X$ such that $f \circ p(x,i) = x$ for every $x \in X$ and $i \in \{0,1\}$.

(b) Prove that Y is Hausdorff if and only if X is Hausdorff and A is a closed subset of X.

(2) Consider the above construction for $X = [0,1]$ and A an arbitrary subset of $[0,1]$.

Prove that Y is compact. Prove that $K = p(X \times \{0\})$ and $L = p(X \times \{1\})$ are compact, and that $K \cap L$ is homeomorphic to A.

We have thus shown that the intersection of two compact subsets in a space that is not Hausdorff may be non-compact and not closed.

Exercise 6. The goal of this exercise is to show *there exists an embedding of the real projective plane* $\mathbb{R}P^2$ *in* $\mathbb{R}^4$.

Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$ given by $\mathbb{S}^2 = \{(x,\, y,\, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, and let $f : \mathbb{S}^2 \to \mathbb{R}^4$ be defined by $f(x,\, y,\, z) = (x^2 - y^2,\, xy,\, yz,\, zx)$.

Prove that

(1) f determines a continuous map $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}^4$ where $\mathbb{R}P^2$ is the real projective plane

(2) $\tilde{f}$ is a homeomorphism onto a topological subspace of $\mathbb{R}^4$.

[*Hint: when proving that* $\tilde{f}$ *is injective you need to show that*

$(x_1^2 - y_1^2,\, x_1y_1,\, y_1z_1,\, z_1x_1) = (x_2^2 - y_2^2,\, x_2y_2,\, y_2z_2,\, z_2x_2) \Rightarrow (x_2,\, y_2,\, z_2) = \pm(x_1,\, y_1,\, z_1)$ *for points* $(x_1,\, y_1,\, z_1)$ *and* $(x_2,\, y_2,\, z_2)$ *in* $\mathbb{S}^2$.

You might try considering the cases $x_1 \neq 0$ *and* $x_1 = 0$.]