STRING THEORY I

Lecture 9

$\boxed{4}$ Interactions

4.1 Generalities

4.2 Vertex operators: introduction

4.3 Vertex operators: open string

- tachyons

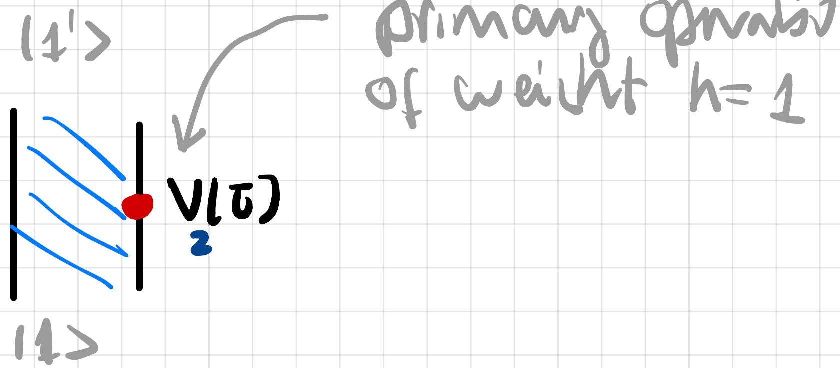
- level 1 & level 2

4.4 The state vertex correspondence

Last lecture:

We introduced vertex operators to describe the emission or absortion of physical string states from the perspective of a fixed string world sheet

For example:



branching of an open string in which an open string in state $|1\rangle$ branches into $|1'\rangle$ & $|2\rangle$

process in which state $|2\rangle$ is emitted (or (absorbed)) at the end point $\sigma=0$ of a fixed string with the vertex $V_2(\tau)$ describing the emission of state $|2\rangle$ from $\sigma=0$ at τ

Open string

- Vertex operator for emission/absorption of a tachyon

$$V_k(\tau) = :e^{ik\cdot X}: \quad \text{has } h = \tfrac{1}{2}(k\cdot k)\ell^2 = 1$$

for the tachyon

$\alpha' k^2 = 1$

where

$$:e^{ik\cdot X}: = e^{k\cdot \sum_{m=1}^{\infty} \frac{\alpha_{-m}}{m} e^{im\tau}} \, e^{ik\cdot x(\tau)} \, e^{-k\sum_{n=1}^{\infty}\frac{\alpha_n}{n} e^{-in\tau}}$$

so we have

level 0 tachyon $\quad |0;k\rangle \xrightarrow{k\cdot k = 2\ell^2} V_k(\tau) = :e^{ik\cdot X(\tau)}:$

$(\alpha' k^2 = 1)$

$h=1$ precisely for $\alpha' k^2 = 1$

$(k^2 = 2\ell)$

Remarks on normal ordering:

- we have adopted above a convention for the normal ordering:

- $:e^{iK\cdot X}:$ vs $e^{iK\cdot X}$

these differ by an exponential $e^{K^2(\cdots)}$

so normal ordering has no effect when $K^2=0$

To see this: note that

$$e^{iK\cdot\alpha_m}\, e^{iK\cdot\alpha_n} = e^{iK\cdot(\alpha_m+\alpha_n)}\, e^{\frac{1}{2}[iK\cdot\alpha_m,\, iK\cdot\alpha_n]}$$

$$= e^{iK\cdot(\alpha_m+\alpha_n)}\, e^{-\frac{1}{2}K\cdot K\, m\,\delta_{m+n,0}}$$

Building vertex operators for level one states

For these states $k^2 = 0$

so we are looking for the photon emission/absorption vertex operator

$$V_s(\tau) = (\cdot \text{- - - -}) \; \underbrace{:e^{ik\cdot X(\tau)}:}$$

- some appropriate local operator
- no zero mode dependence
  ↳ consider $\partial_\tau X^\mu$

as $k^2 = 0$: • $h = 0$
• no ordering issue

$$[L_m, \partial_\tau X^\mu(\tau)] = \frac{\partial}{\partial\tau}[L_m, X^\mu(\tau)]$$

$$= \frac{\partial}{\partial\tau}\left(-i\left(\partial_\tau X^\mu(\tau)\right)e^{im\tau}\right)$$

$$= e^{im\tau}\left(-i\,\partial_\tau + m\right)\partial_\tau X^\mu(\tau)$$

Then $\partial_\tau X^\mu(\tau)$ is primary of weight $h=1$

We could try

polarization

$$V_\zeta(\tau) = (\zeta\cdot\partial_\tau X(\tau)) :e^{ik\cdot X(\tau)}:$$

and deal with normal ordering issues.

In $\zeta \cdot \partial_\tau X(\tau)$: each oscillator operator is contracted with ζ

In $e^{iK\cdot X(\tau)}$: each oscillator operator is contracted with K

Moreover:

$$[\alpha_m \cdot \zeta, \alpha_n \cdot K] = m\, \delta_{m+n,0}\, (\zeta \cdot K)$$

$\therefore$ when $\zeta \cdot K = 0$ (physical polarization) the operator product

$$(\zeta \cdot \partial_\tau X(\tau)) :e^{iK\cdot X(\tau)}: \text{ is well defined}$$

(no corrections from ordering ...)

Then we have:

$$V_\xi(\tau) = (\xi \cdot \dot{X}(\tau)) :e^{iK\cdot X(\tau)}: \text{ with } K^2=0,\ \xi\cdot K=0 \quad (n=1)$$

Note that

$$(K\cdot \dot{X}) :e^{iK\cdot X}: = -i\,\partial_\tau (e^{iK\cdot X})$$

which vanishes after integrating over τ.

This means that the longitudinal mode decouples

- It is not a coincidence that the rules for constructing vertex operators is closely parallel to the construction of physical states.

Here we have for level 1

vertex operators with $n=1$ $\longleftrightarrow$ physical states

Building vertex operators for level two states

Harder. At level two we guess

$$\gamma_{\mu\nu}\,\alpha^{\mu}_{-1}\,\alpha^{\nu}_{-1}\,|0;k\rangle \longleftrightarrow \gamma_{\mu\nu}\,\dot{X}^{\mu}\,\dot{X}^{\nu}\,:\underbrace{e^{ik\cdot X}}_{h=-1}:$$

mass-shell : $\ell^2 k^2 = -2$

for $\ell^2 k^2 = -2$

normal ordering correction:

$\gamma_{\mu\nu}\,\dot{X}^{\mu}\,\dot{X}^{\nu}$ oscillator reorderings are proportional to $\gamma^{\mu}{}_{\mu}$ so require traceless γ

$\gamma_{\mu\nu}\,\dot{X}^{\mu}\,\dot{X}^{\nu}$ & $:e^{ik\cdot X}:$ oscillator reorderings proportional to $k\cdot\gamma$ so require $k\cdot\gamma = 0$ ie transverse polarization

$$\gamma_{\mu\nu}\,\alpha_{-1}^{\mu}\,\alpha_{-1}^{\nu}\,|0;k\rangle \longleftrightarrow \gamma_{\mu\nu}\,\dot X^{\mu}\,\dot X^{\nu} :\underbrace{e^{ik\cdot X}}_{h=-1 \quad (\ell^2 k^2 = -2)}:$$

with $k\cdot\gamma = 0$ & γ traceless

These are the conditions for a massive spin two state, that is a symmetric traceless 2-tensor of $SO(D-1) = SO(25)$

It is the only possibility in $D=26$ (see GSW)

$\ddot X^{\mu} : e^{ik\cdot X}:$??

As a consequence we have a map

state $\longrightarrow$ operator

in which $|\phi\rangle_{\text{open}} \longmapsto V_\phi(\tau)$ conformal primary of dimension $h=1$

For example: noting that $:e^{ik\cdot X}:$ has $h=\frac{1}{2}(k\cdot k)$

level 0 tachyon $|0;k\rangle \longmapsto :e^{ik\cdot X(\tau)}:$

$\ell^2 k\cdot k=2$

level 1 photon $|\xi;k\rangle \longmapsto :(\xi\cdot\dot{X}(\tau))e^{ik\cdot X(\tau)}:$ $\xi\cdot k=0$

$k\cdot k=0$

level 2 massive spin 2 $|\gamma;k\rangle \longmapsto :\gamma_{\mu\nu}\dot{X}^\mu(\tau)\dot{X}^\nu(\tau)e^{ik\cdot X(\tau)}:$ $\gamma^\mu{}_\mu=0$

$\ell^2 k\cdot k=-2$ $k\cdot\gamma=0$

Gauge invariance:

level 1: for the photon vertex operator with $\xi \sim k$

(longitudinal mode)

$$\int d\bar{\tau} \; k\cdot\dot{X}\, e^{ik\cdot X} = -i\int d\bar{\tau}\; \partial_{\bar{\tau}}\left(e^{ik\cdot X}\right) = 0$$ up to boundary terms

so longitudinal mode decouples.

4.4 State-vertex correspondence

It is not a coincidence that there is a similarity between states & vertex operators.

Note that if $V(\tau)$ is a vertex operator describing the emission (or absorption) from the end point $\sigma = 0$ at some time τ, then

$$V(\tau) = e^{i\tau L_0} V(0) e^{-i\tau L_0}$$

because the Hamiltonian is $L_0 - a = L_0 - 1$.

For the tachyon:

$$V_T(k,\tau) = :e^{ik\cdot X(\tau)}: = e^{i\tau L_0} \underbrace{:e^{ik\cdot X(0)}:}_{V_T(k,0)} e^{-i\tau L_0}$$

Consider the action of this operator on the zero momentum string vacuum state $|0;0\rangle$

$L_0 = \frac{\ell^2}{2} p^2 + N$

$$V_T(k,\tau)|0;0\rangle = e^{i\tau L_0} :e^{ik\cdot X(0)}: e^{-i\tau L_0} |0;0\rangle$$

$$= e^{i\tau L_0} :e^{ik\cdot X(0)}: |0;0\rangle \quad \text{as } L_0|0;0\rangle = 0$$

$$= e^{i\tau L_0} e^{k\cdot \sum_{n=1}^{\infty} \frac{\alpha_{-m}}{m}} :e^{ik\cdot x(0)}: e^{-k\sum_{n=1}^{\infty}\frac{\alpha_n}{n}} |0;0\rangle$$

$$= e^{i\tau L_0} e^{k\cdot \sum_{m\geq 1} \frac{1}{m}\alpha_{-m}} \underbrace{:e^{ik\cdot x(0)}: |0;0\rangle}_{|0;k\rangle} \quad \text{as } \alpha_n|0;0\rangle = 0 \;\; \forall\, n \geq 1$$

$$V_T(K,\tau)\,|0;0\rangle = e^{i\tau L_0}\, e^{K\cdot\sum_{m\geq 1}\frac{1}{m}\alpha_{-m}}\,|0;K\rangle$$

Define $z = e^{i\tau} = e^{t}$, where $\tau = -it$ (so t is Euclidean world sheet time). Then

$L_0 = \frac{1}{2}p^2 + N$

$$V_T(K,\tau)\,|0;0\rangle = z^{L_0}\, e^{K\cdot\sum_{m\geq 1}\frac{1}{m}\alpha_{-m}}\,|0;K\rangle$$

$$= z^{1+N}\left(1 + K\cdot\alpha_{-1} + \frac{1}{2}\left((K\cdot\alpha_{-2}) + (K\cdot\alpha_{-1})^2\right) + \cdots\right)|0;K\rangle$$

$z^{L_0}(\cdots)|0;K\rangle = e^{\frac{1}{2}K^2+N}(\cdots)|0;K\rangle, \quad . \quad \frac{1}{2}K^2 = 1$

$$V_T(K,\tau)|0;0\rangle = z\left(1 + z\,K\cdot\alpha_{-1} + \frac{1}{2}z^2\left((K\cdot\alpha_{-2}) + (K\cdot\alpha_{-1})^2\right) + \mathcal{O}(z^3)\right)|0;K\rangle$$

Then we can recover the state $|0;K\rangle$ from V_T by taking

$$|0;K\rangle = \lim_{z\to 0}\frac{1}{z}V_T(K;\tau)|0;0\rangle = \lim_{t\to -\infty} e^{-t}\, V_T(K;it)\,|0;K\rangle$$

For the photon

$$V_\varsigma(K,\tau) = \varsigma \cdot \dot{X}\, e^{iK\cdot X} \quad |0;0\rangle = z^N \sum_{m>0} (\varsigma\cdot\alpha_{-m})\, e^{\sum_{n>0}\frac{1}{n}K\cdot\alpha_{-n}} \quad |0;K\rangle$$

$$= \Big(z\,(\varsigma\cdot\alpha_{-1}) + z^2\big((\varsigma\cdot\alpha_{-2}) + (\varsigma\cdot\alpha_{-1})(K\cdot\alpha_{-1})\big) + \cdots \Big)\,|0;K\rangle$$

We recover the state $|\varsigma;K\rangle$ from

$$\lim_{z\to 0} \frac{1}{z} V_\varsigma(K;\, it)\,|0;0\rangle$$

Continuing like this, a pattern becomes clear:

for a physical state $|\psi\rangle$ with vertex operator $V_\psi(\tau)$ we have

$$|\psi\rangle = \lim_{z\to 0} \frac{1}{z} V_\psi(it)\,|0;0\rangle$$

An analogous statement holds for "out" states. For example

$$\langle 0;0|\,V_T(K;it) = \langle 0;K|\, e^{\sum_{n>0}\frac{1}{n}\alpha_n\cdot K}\, z^{-L_0}$$

$$= \frac{1}{z}\langle 0;K|\left(1 + z^{-1}(\alpha_1\cdot K) + \frac{1}{2} z^{-2}\left(\alpha_2\cdot K + (\alpha_1\cdot K)^2\right) + \cdots\right)$$

$$\langle 0;K| = \lim_{z\to 0} z\,\langle 0;0|\,V_T(K,\tau) = \lim_{t\to\infty} e^{t}\,\langle 0;0|\,V_T(K;it)$$

$$\boxed{\begin{aligned} |\psi\rangle &= \lim_{z\to 0} \frac{1}{z} V_\psi(it)\,|0;0\rangle \qquad &\text{in} \\ \langle\psi| &= \lim_{z\to\infty} z\,\langle 0;0|\,V_\psi(it) \qquad &\text{out} \end{aligned}}$$

One can also use this to describe an incoming state $|\psi\rangle$ (or an outgoing state $\langle\psi|$) by acting with the vertex operator on the two momentum vacuum state in the infinite Euclidean past ie $t\to-\infty$ (resp Euclidean future, $t\to\infty$)

The picture presented here is part a general treatment of the operator-state correspondence in conformal field theory where the general construction is

$$A(\tau) \longrightarrow |\psi_A\rangle = \lim_{t\to-\infty} z^{h_A} A(it)|\Omega\rangle$$

$$\longrightarrow \langle\psi_A| = \lim_{t\to\infty} z^{-h_A} \langle\Omega|A(it)$$

where $|\Omega\rangle$ is the vacuum state.

Next:

→ Vertex operators for the closed string

→ 3 & 4 point interactions

end of lecture 9