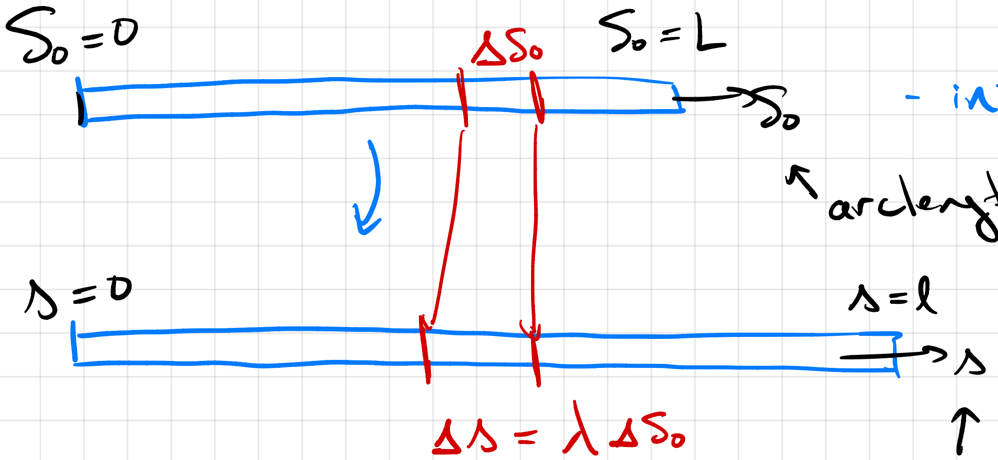
Bio Growth

Motivation – all bio entities grow

– growth very much a mechanical process

1D Growth – We consider a 1D body (rod) constrained to a line that deforms due to growth (increase of mass) and/or elastic response (stretching / compressing)



– initial configuration

arclength in init. state

– current configuration

current arclength

We define

$$\lambda := \frac{\partial s}{\partial S_0} = \lambda(S_0)$$

is stretch from init. to current

($\lambda > 0 \Rightarrow$ 1-1 map b/t S_0 and s)

Purely elastic deformation (No growth)

$\lambda = \alpha$. Let σ be axial stress in rod
call
(force / area of section, A)

Define $n = \sigma A$ ($= n_3$ from Biofilaments)

Force balance: $\dfrac{dn}{dS_0} + f = 0$

f: body force per unit (initial) length

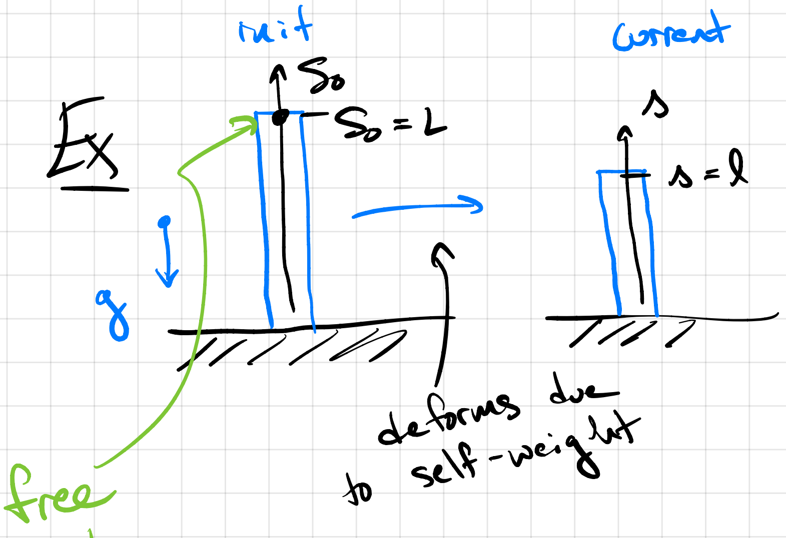
Constitutive law: $n = h(\alpha)$

w/ $h(1) = 0$

eg Hooke's Law: $h(\alpha) = EA(\alpha - 1)$

Young's mod

Ex



$f = -\rho g$

so $n' = \rho g$, $\left(' = \dfrac{d}{dS_0} \right)$

w/ $n(L) = 0$

$$\rightarrow \boxed{n = \rho g (S_0 - L)}$$

$$n = EA(\alpha - 1) \Rightarrow \alpha = \frac{\rho g}{EA}(S_0 - L) + 1$$

And $\alpha = \dfrac{ds}{dS_0} \Rightarrow s = \int \alpha \, dS_0$

$$\Rightarrow \ell = \int_0^L \alpha \, dS_0 = L + \frac{\rho g L^2}{2EA} - \frac{\rho g L^2}{EA} = L - \frac{\rho g L^2}{2EA}$$

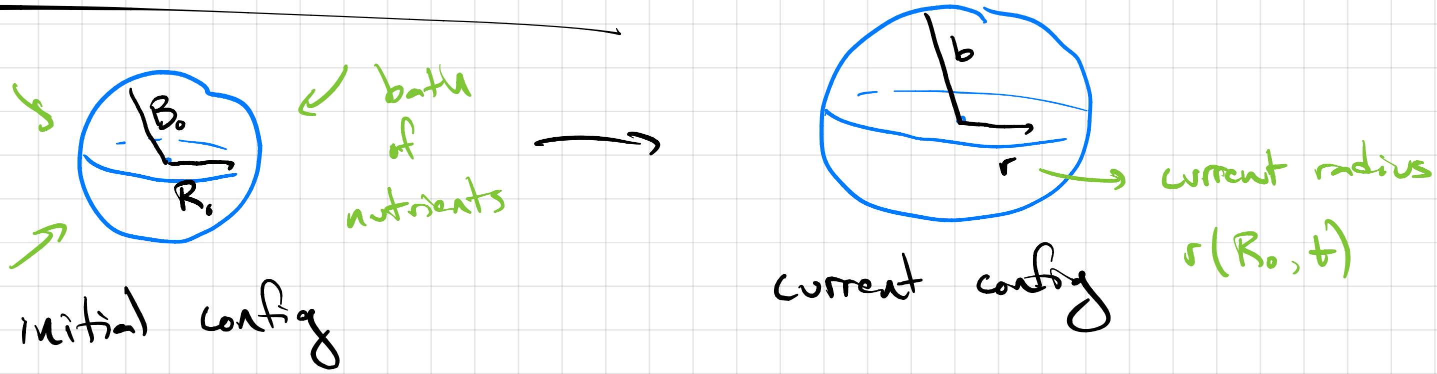
Pure Growth deformation (No elasticity)

$\lambda = \gamma$ · Growth process : $\gamma = \gamma(t)$ follows
call a growth law : $\dfrac{\partial \gamma}{\partial t} = G(\gamma, s, S_0, \ldots)$

eg $\dfrac{\partial \gamma}{\partial t} = k\gamma \rightarrow \gamma = e^{kt}$ $(\gamma \equiv 1$ at $t=0)$

& $\dfrac{\partial s}{\partial S_0} = \gamma \Rightarrow s = S_0 e^{kt}$ $(s(0,t) = 0)$

Application - tumour spheroid



Assume i) Isotropic growth (same in all directions), exponential in time, proportional to nutrient concent. $u(r,t)$

ii) Nutrient diffuses in from bath, uptake Q by spheroid

iii) Constant nutrient concent. at outer surface

Define volumetric growth by $dv = \eta \, dV_0$

current volume element · init. vol. element

For sphere, $dv = r^2 \sin\varphi \, d\varphi \, d\theta \, dr$, $dV_0 = R_0^2 \sin\varphi \, d\varphi \, d\theta \, dR_0$

$$\Rightarrow r^2 dr = \eta R_0^2 dR_0 \Rightarrow \frac{dr}{dR_0} = \underbrace{\eta \left(\frac{R_0}{r}\right)^2}_{\gamma}$$

Growth: $\frac{\partial \eta}{\partial t} = K\eta\, u(r,t)$ $\quad (\eta = \eta(r,t))$

Nutrient: $\frac{\partial u}{\partial t} = D\nabla^2 u - \underline{Q} = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) - Q$

Diffusion const   uptake   little r since diff. occurs in current state

w/ $u(b,t) = u_b$ (const)

- "Fast diffusion" $u_t \approx 0$ (equil for nutrient)

- idea is to solve for $u(r,t)$, then insert into

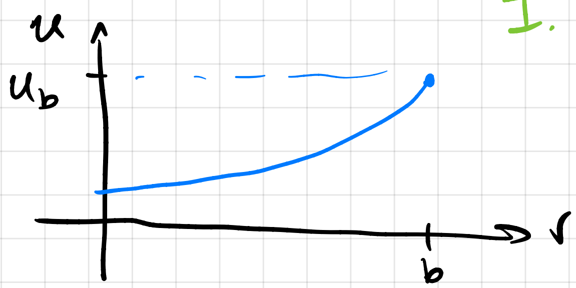
$$\frac{\partial \eta}{\partial t} = k\eta u, \quad \frac{\partial r}{\partial R_0} = \eta \frac{R_0^2}{r^2}$$

We set $\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{Q}{D} r^2 \Rightarrow \frac{\partial u}{\partial r} = \frac{Qr}{3D} + \frac{c_1}{r^2}$

$$\Rightarrow u = \frac{Qr^2}{6D} - \frac{c_1}{r} + c_2$$

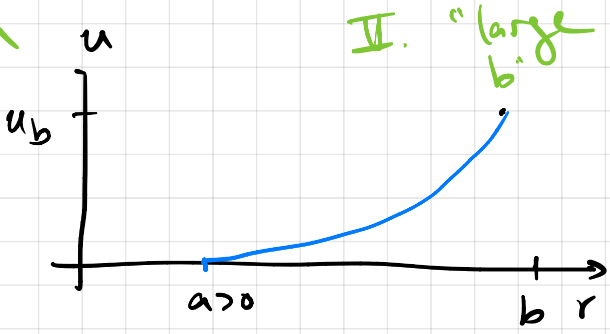
Key point: Must have $u \geq 0 \Rightarrow$ can have either

I. "small enough b"



OR

II. "large b"



I. Set $c_1 = 0$, find c_2 via $u(b,t) = u_b$

$$\rightarrow u = \frac{Q}{6D}(r^2 - b^2) + u_b \overset{\text{call}}{=} u_I(r;b)$$

Switch to Case II when $u_I(0;b) = 0$

$$\rightarrow b_{crit} = \left(\frac{6Du_b}{Q}\right)^{1/2}$$

Case II: $b > b_{crit}$, $u(b,t) = u_b$, $u(a,t) = 0$

$$\rightarrow u_{II} = \begin{cases} 0 & r < a \\ \frac{Qr^2}{6D} + Q(b^3 - a^3) \ldots & a < r < b \end{cases}$$

"necrotic core"

(in typed notes)

- a is det'd from setting $\frac{\partial u}{\partial r}(a,t) = 0$

$\rightsquigarrow$ polynomial $\frac{Q}{D}\left(2a^3 - 3a^2 b + b^3\right) - 6bu_b = 0$

Back to Growth ...

$$r^2\,dr = \eta R_0^2\,dR_0 \Rightarrow \int_0^b r^2\,dr = \int_0^{B_0} \eta R_0^2\,dR_0$$

$$\Rightarrow \frac{b^3}{3} = \int_0^{B_0} \eta R_0^2\,dR_0$$

we want to use $\frac{\partial \eta}{\partial t} = k\eta u$

$$\xRightarrow{\frac{\partial}{\partial t}} b^2 \frac{db}{dt} = \int_0^{B_0} \eta_t R_0^2\,dR_0 = k\int_0^{B_0} u \underbrace{\eta R_0^2\,dR_0}_{r^2 dr}$$

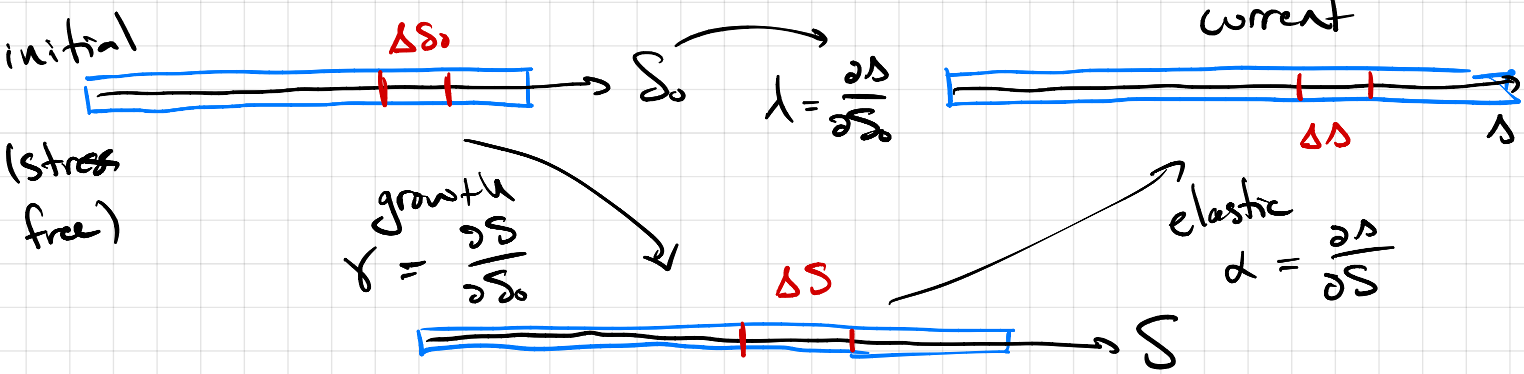
$$= k\int_0^b u(r,t)\, r^2\,dr$$

so the outer radius $b = b(t)$ evolves according to

$$\boxed{b^2 \frac{db}{dt} = k\int_0^b u(r,t;b)\, r^2\,dr}$$

solve w/ $b(0) = B_0$, and use $u = u_I$ when $b < b_{crit}$, use $u = u_{II}$ when $b > b_{crit}$

Growth with Elastic Response



reference config (stress free)

Decomposition

$$\lambda = \frac{\partial s}{\partial S_0} = \frac{\partial s}{\partial S}\,\frac{\partial S}{\partial S_0} = \alpha \gamma$$

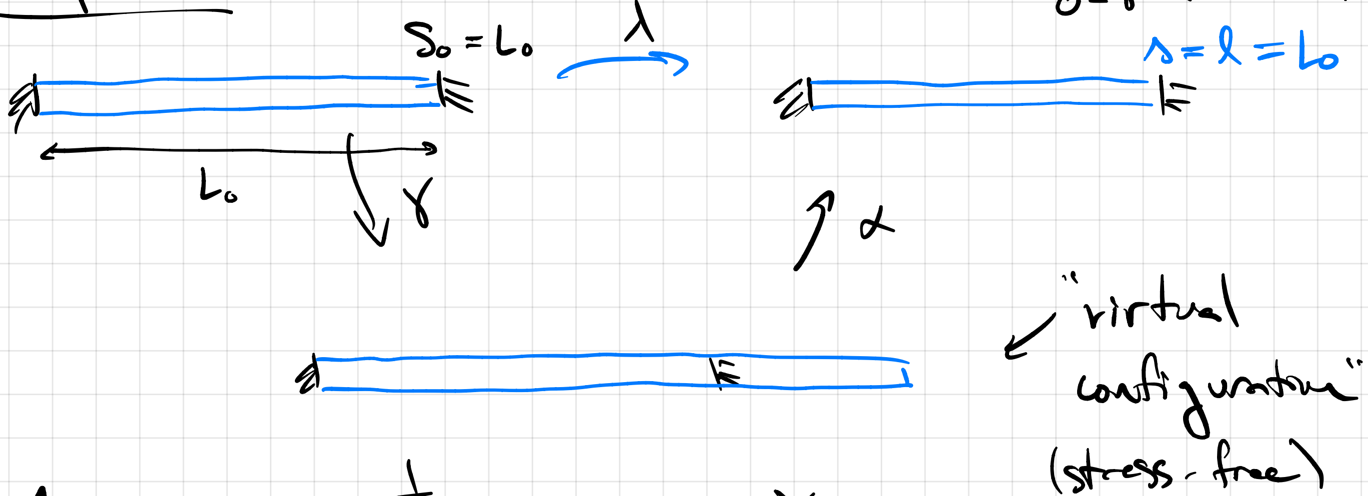
Note: require $\alpha, \gamma > 0$

$\gamma > 1$: Growth

$0 < \gamma < 1$: resorption

"1D Morphoelasticity"

Simple Ex – A rod between walls



$\lambda \equiv 1 \Rightarrow \alpha = \frac{1}{\gamma}$. Supp. $\gamma = 1 + t$

Hookean: Stress $\sigma = E(\alpha - 1) = \frac{-Et}{1+t}$

But as $t \to \infty$, $\sigma \to -E$. infinite compression but only finite stress !!

Better: neoHookean $\sigma = \frac{E}{3}\left(\alpha^2 - \frac{1}{\alpha}\right) = \frac{E}{3}\left(\frac{1}{(1+t)^2} - (1+t)\right)$

Now $\sigma \sim -\frac{E}{3} t$ as $t \to \infty$

Stress-dependent Growth

$$\frac{\partial \gamma}{\partial t} = \kappa \gamma (\sigma - \sigma^*)$$

σ^* (const) is called homeostatic (target) stress

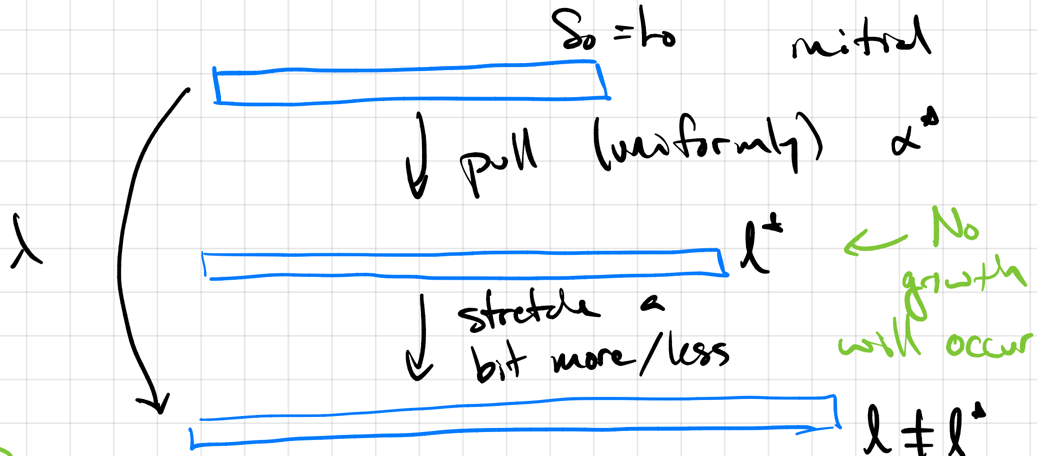
Ex. Hookean material $\sigma = E(\alpha - 1)$

Homeostatic state: $\sigma = \sigma^* \Rightarrow \alpha = \frac{\sigma^*}{E} + 1 \overset{\text{call}}{=} \alpha^*$

- so if no growth, $\gamma = 1$, $S = S_0$, then

$$\lambda = \alpha = \alpha^* = \frac{\partial s}{\partial S_0} \Rightarrow l^* = \alpha^* L_0$$

An experiment



Grows to (try to) recover σ^*

Now hold

$$\lambda = \frac{l}{L_0} = \frac{l}{l^*} \cdot \frac{l^*}{L_0} = \frac{l}{l^*} \alpha^*$$

Now, $\alpha = \frac{\lambda}{\gamma} = \frac{l}{l^* \gamma} \cdot \alpha^* = \frac{l}{l^* \gamma} \cdot \left(\frac{\sigma^*}{E} + 1 \right)$

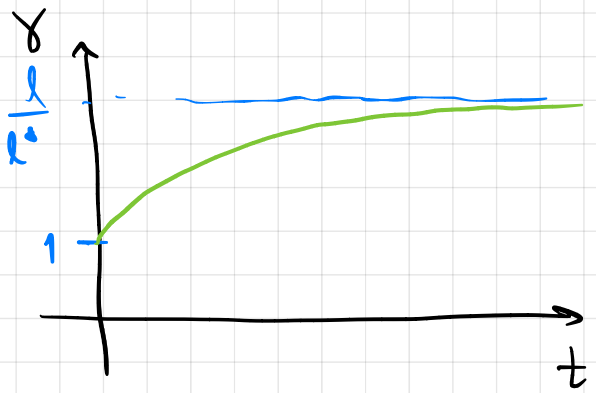
$$\rightarrow \sigma = E(\alpha - 1) \Rightarrow \dot{\gamma} = \gamma(\sigma - \sigma^*) = \frac{l}{l^*}(\sigma^* + E) - (\sigma^* + E)\gamma$$

$\kappa = 1$

Solve w/ $\gamma(0) = 1$,

we get

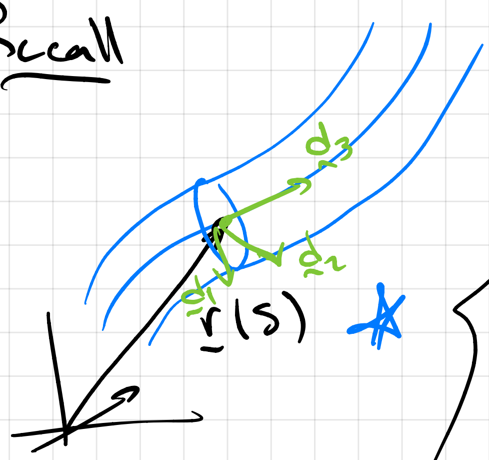
$$\boxed{\gamma(t) = \frac{l}{l^*} + \left(1 - \frac{l}{l^*}\right) e^{-(\sigma^* + E)t}}$$



A Growing Elastic Rod

Idea Extend the framework of elastic rods to include growth

Recall



$$\frac{\partial \underset{\sim}{r}}{\partial S} = \alpha \underset{\sim}{d}_3$$ unshearable, extensible

$$\frac{\partial \underset{\sim}{d}_i}{\partial S} = \underset{\sim}{u} \wedge \underset{\sim}{d}_i$$

$$\frac{\partial \underset{\sim}{n}}{\partial S} + \underset{\sim}{f} = \underset{\sim}{0} \qquad \text{(FB)}$$

$$\frac{\partial \underset{\sim}{m}}{\partial S} + \frac{\partial \underset{\sim}{r}}{\partial S} \wedge \underset{\sim}{n} + \underset{\sim}{l} = \underset{\sim}{0} \qquad \text{(MB)}$$

Plus constitutive:

$$\star\star \begin{cases} \underset{\sim}{m} = EI_1(u_1 - \hat{u}_1)\underset{\sim}{d}_1 + EI_2(u_2 - \hat{u}_2)\underset{\sim}{d}_2 + \mu J(u_3 - \hat{u}_3)\underset{\sim}{d}_3 \\ \underset{\sim}{n} \cdot \underset{\sim}{d}_3 = n_3 = EA(\alpha - 1) \end{cases}$$

Above, $\alpha = \frac{\partial s}{\partial S}$ where S - ref. arclength
s - current arclength

To incorporate <u>axial</u> <u>growth</u>, we introduce a fixed <u>initial</u> config w/ arclength S_0, a <u>grown</u> ref config w/ arclength S, such that

$\gamma = \frac{\partial S}{\partial S_0}$ is <u>growth stretch</u>.

As before, total stretch $\lambda = \frac{\partial s}{\partial S_0} = \alpha \gamma$

Eq'ns ☆, ☆☆ don't change, but the domain $S \in [0, L]$ will change for $\gamma \neq 1$

- Can also cast the eq'ns ito S_0 or s

eg in S_0 we'd write $\frac{\partial \underset{\sim}{r}}{\partial S_0} = \alpha \gamma \underset{\sim}{d}_3$,

$$\frac{\partial \underset{\sim}{n}}{\partial S_0} + \gamma \underset{\sim}{f} = \underset{\sim}{0} \quad , \text{etc.} \ldots$$

[Assumed in ☆ that $\underset{\sim}{f} = \frac{\text{Force}}{\text{Unit ref length } \Delta S}$

$$\Rightarrow \gamma \underset{\sim}{f} = \frac{\text{Force}}{\text{Unit init length } \Delta S_0} \Big]$$

- Cross-sectional growth would only change the parameters I_1, I_2, J, A - and thus only impact stiffness

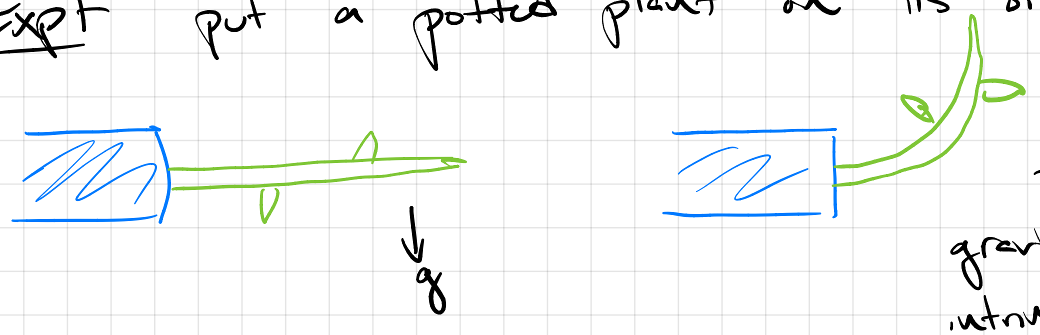
eg. circular cross-section w/ radius $R = \varepsilon(1+t)$

$$\rightsquigarrow I_1 = I_2 = \frac{\pi R^4}{4} \sim \varepsilon t^4, \quad A = \pi R^2 \sim \varepsilon t^2$$

Remodelling - A change in properties without a change in mass. - eg a change in $\hat{\underset{\sim}{u}}$ (intrinsic curvature)

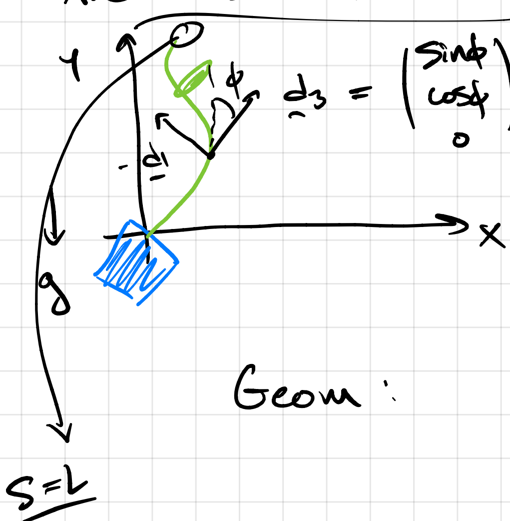
Application Gravitropism

Expt put a potted plant on its side



the plant "wants to be" aligned w/ gravity → develops intrinsic curvature

An elastic rod model



$\underset{\sim}{d}_3 = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$, $\underset{\sim}{d}_1 = \begin{pmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{pmatrix}$, $\underset{\sim}{d}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\underset{\sim}{d}_3' = \phi' \underset{\sim}{d}_1$

$\downarrow$

$\underset{\sim}{u} = u_2 \underset{\sim}{d}_2 = \phi' \underset{\sim}{d}_2$

Geom: $x' = \sin\phi$

$y' = \cos\phi$

$\phi' = u_2$

$\underset{\sim}{n} = n_x \underset{\sim}{e}_x + n_y \underset{\sim}{e}_y$

FB $\begin{cases} n_x' = 0 \\ n_y' = \rho g \end{cases}$

$\underset{\sim}{f} = -\rho g \underset{\sim}{e}_y$

Plus BC $\Rightarrow$ $n_x \equiv 0$, $n_y = \rho g (s-L)$

MB $m' + \sin\phi \, \rho g (s-L) = 0$

$\underset{\sim}{m} = m \underset{\sim}{d}_2$

CL $m = EI(u_2 - \hat{u}_2)$

At $s=0$: $x=y=0$, $\phi = \phi_0 \leftarrow$ pot angle

Gravitropism: $\boxed{\dfrac{\partial \hat{u}_2}{\partial t} = -\beta \sin\phi}$

Quasistatic Evolution

Kinematics Only

$u_2 \equiv \hat{u}_2 \qquad (\underset{\sim}{n} \equiv \underset{\sim}{m} \equiv \underset{\sim}{0})$

- if nearly vertical, $|\phi| \ll 1$:

$$\begin{cases} x' \approx \phi \\ y' \approx 1 \\ \phi' = u_2 \\ \dot{u}_2 \approx -\beta \phi \end{cases} \longrightarrow x_{sst} + \beta x_s = 0$$

$$\longrightarrow x_{st} + \beta x = c(t)$$

BC At $s=0$, $x=0$, $x_s(0,t)=\phi_0$

$$\downarrow$$

$$x_{st}(0,t)=0$$

$$\Rightarrow c(t)=0$$

We are left w/ :

$$\begin{cases} x_{st} + \beta x = 0 & (1) \\ x(0,t) = 0 & (2) \\ x(s,0) = \phi_0 s & (3) \end{cases}$$

if straight, ie $\phi \equiv \phi_0$, at $t=0$

Can solve as similarity sol'n (BS.2!)

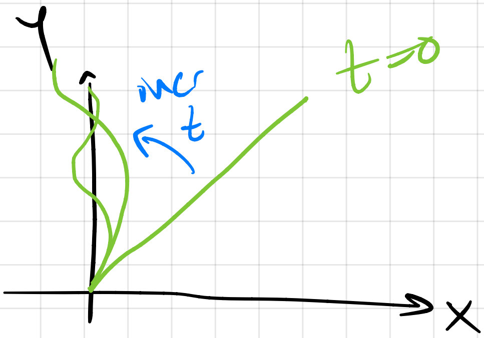
- seek $x(s,t) = s^{\alpha} f(\underbrace{st}_{\text{call } \eta})$

$(3) \Rightarrow \alpha=1$, $f(0)=\phi_0$.

$(1) \rightarrow \eta f''(\eta) + 2f'(\eta) + \beta f = 0$

$$\leadsto \text{sol'n} \quad f = \phi_0 \frac{J_1(2\sqrt{\beta\eta})}{\sqrt{\beta\eta}}$$

J_1 Bessel fn of 1st kind



Mechanical Pattern Formation

"Simple State" $\longrightarrow$ Complex State (Patterned)

Growth + constraints

Compare

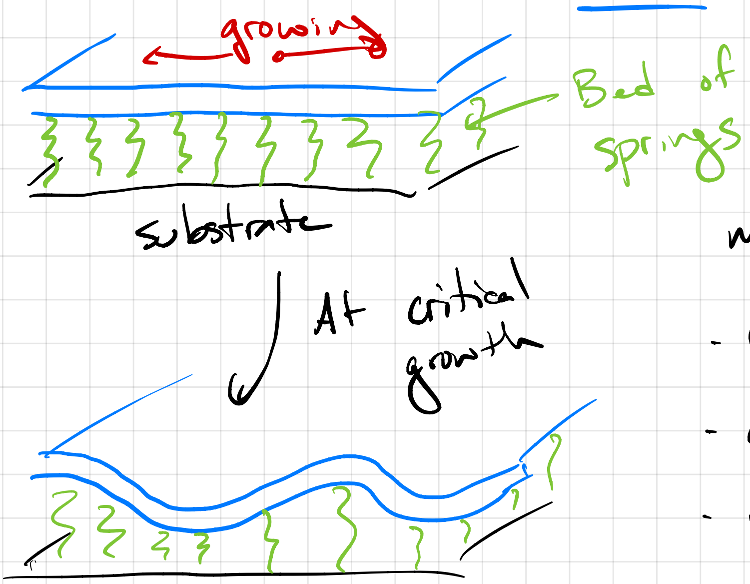
(A) in a biochemical pattern (Turing pattern), concentrations of chemicals go from a homogeneous state to patterned state due to reaction, diffusion

(B) A biomechanical pattern is structural, ie a material deforms from a "simple" base state (eg flat) to a patterned state.

- Note: both types may be present, & linked!

Ex Wrinkling instability - Rod on Foundation model

Ingredients: a growing elastic beam (or sheet) $\rightarrow$ extensible
attached to a substrate (foundation)

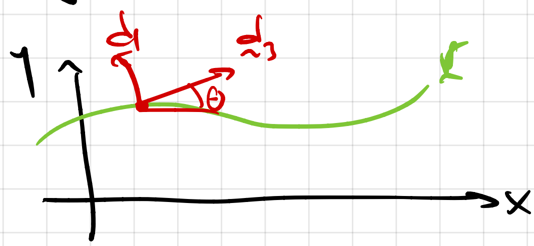


Basic idea underlies morphogenesis of many patterns
- wrinkles in skin
- cortical folds in brain
- ridges / spines in seashells
- shape of airways / intestine

Growing beam

$$\underset{\sim}{r}(S_0) = x(S_0)\,\underset{\sim}{e}_x + y(S_0)\,\underset{\sim}{e}_y$$

- $$\frac{d\underset{\sim}{r}}{dS_0} = \alpha\gamma\, \underset{\sim}{d}_3$$



$$\underset{\sim}{d}_3 = \cos\theta\, \underset{\sim}{e}_x + \sin\theta\, \underset{\sim}{e}_y$$

$$\underset{\sim}{d}_1 = -\sin\theta\, \underset{\sim}{e}_x + \cos\theta\, \underset{\sim}{e}_y$$

- $$\underset{\sim}{d}_3'(S_0) = \theta'(S_0)\, \underset{\sim}{d}_1$$

$$\underset{\sim}{u} = u_2 \underset{\sim}{d}_2 = \frac{d\theta}{dS}\underset{\sim}{d}_2, \quad (\underset{\sim}{d}_2 = \underset{\sim}{e}_z)$$

$$= \frac{1}{\gamma}\theta'(S_0)\, \underset{\sim}{e}_z$$

Define $\underset{\sim}{n} = n_x \underset{\sim}{e}_x + n_y \underset{\sim}{e}_y$, $\underset{\sim}{m} = m \underset{\sim}{e}_z$

<u>FB</u> $\rightarrow$ $\dfrac{dn_x}{dS_0} + f = 0$, $\dfrac{dn_y}{dS_0} + g = 0$

where $\underset{\sim}{f} = f\underset{\sim}{e}_x + g\underset{\sim}{e}_y$ is force due to substrate.

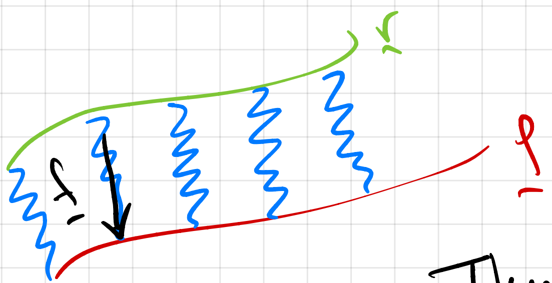
<u>MB</u> $\rightarrow$ $\dfrac{dm}{dS_0} + \alpha\gamma\left(n_y \cos\theta - n_x \sin\theta\right) = 0$

<u>Constitutive laws</u> $m = EI u_2 = \dfrac{EI}{\gamma}\,\theta'(S_0)$

$n_3 = \underset{\sim}{n} \cdot \underset{\sim}{d}_3 = EA(\alpha - 1) \Rightarrow n_x \cos\theta + n_y \sin\theta = EA(\alpha - 1)$

<u>The foundation</u> - define a curve $\underset{\sim}{\rho}(S_0) = \rho_x \underset{\sim}{e}_x + \rho_y \underset{\sim}{e}_y$.

and create a 1-1 map "gluing" $\underset{\sim}{r}$ to $\underset{\sim}{\rho}$ using elastic springs



let $\Delta(S_0) = \|\underset{\sim}{r}(S_0) - \underset{\sim}{\rho}(S_0)\|$,

Then $\underset{\sim}{f} = h(\Delta - \delta)\left(\dfrac{\underset{\sim}{\rho} - \underset{\sim}{r}}{\Delta}\right)$

rest length

- h describes strength/properties of attachment
  - should satisfy $h(0) = 0$, $h'(0) > 0$

<u>Simplest</u> $\underset{\sim}{\rho} = S_0 \underset{\sim}{e}_x$, and supp. before growth,

$\underset{\sim}{r} = S_0 \underset{\sim}{e}_x$ $(\delta = 0)$.

- If $h(0) = 0$, $h'(0) = \kappa$, then

$$\underset{\sim}{f} \approx -\kappa\left((x - S_0)\underset{\sim}{e}_x + y\,\underset{\sim}{e}_y\right)$$

$$\Rightarrow \quad \frac{dn_x}{dS_0} = \kappa(x - S_0), \quad \frac{dn_y}{dS_0} = \kappa y$$

Observe it is possible to have growth $(\gamma > 1)$ without any deformation:

- take $\gamma > 1$, and $\lambda = 1 = \alpha\gamma \Rightarrow \alpha = \frac{1}{\gamma}$

$x = S_0$, $y = 0$, $\theta = 0$, $m = 0$, $n_y = 0$,

$$n_x = EA(\alpha - 1) = EA\left(\frac{1}{\gamma} - 1\right)$$

- Compressed but still flat

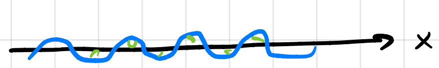
Pattern forms when compressive energy gets too high $\rightarrow$ a tradeoff of bending energy & spring (foundation) energy to relieve some compressive energy

When? What kind of pattern?

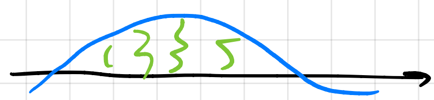
- depends on material parameters. EI EA
  & substrate properties κ

Compare:

substrate >> bending



bending >> substrate



Buckling analysis:

$$x = s_0 + \varepsilon x_1$$

$$\theta = \varepsilon \theta_1$$

$$\alpha = \frac{1}{\gamma} + \varepsilon \alpha_1$$

$\vdots$

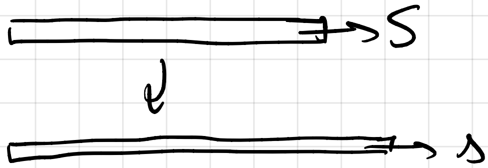
Find crit γ (eig value) at which linearised system has a soln.

3D Growth

1. Review / Summary of nonlinear elasticity
2. Build in growth $\rightsquigarrow$ Morphoelasticity

Nonlinear Elasticity in 10 Easy Steps

(0.) Recall 1D :

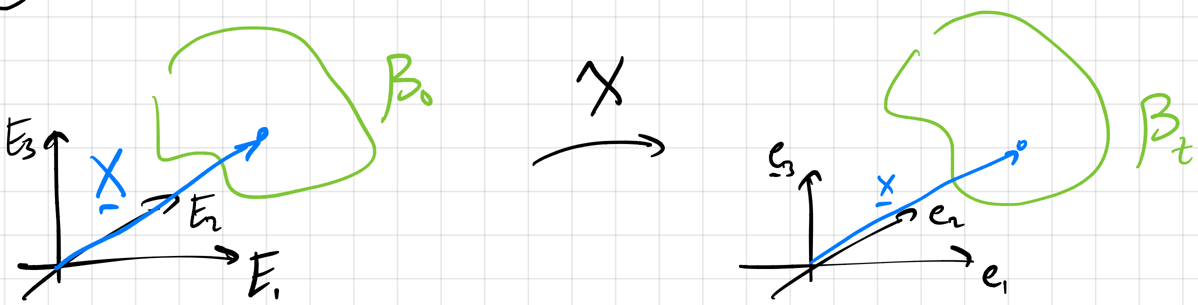


$\frac{ds}{dS} = \alpha$ strain

$n'(s) + f = \rho \ddot{s}$ FB

$n = \hat{n}(\alpha)$ CL

(1) Kinematics . continuous deformation of body $\mathcal{B}_0$ to $\mathcal{B}_t$



$$\mathbf{x} = \chi(\mathbf{X}, t)$$

(2) Tensors $\quad \mathbf{u} \otimes \mathbf{v}$ tensor product defined by

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u}$$

A tensor $T = T_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$, $\quad T_{ij} = T\mathbf{e}_j \cdot \mathbf{e}_i$

Let $\phi(\mathbf{x})$ be scalar, $\mathbf{u} = u_i \mathbf{e}_i$ a vector, $T = T_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$ a tensor

Then $\operatorname{grad}\phi = \frac{\partial \phi}{\partial x_i} \mathbf{e}_i$, $\quad \operatorname{grad}\mathbf{u} = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$.

$\operatorname{div} T = \frac{\partial T_{ij}}{\partial x_i} \mathbf{e}_j$. $\quad$ If $A = A(F)$, $\quad \frac{\partial A}{\partial F} = \frac{\partial A}{\partial F_{ji}} \mathbf{e}_i \otimes \mathbf{e}_j$

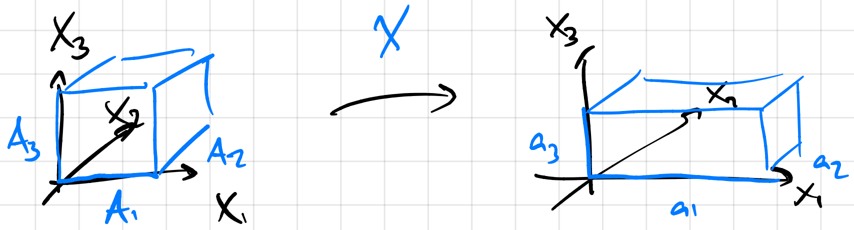
scalar $\quad$ tensor

③ Deformation Gradient Tensor

$$F = \operatorname{Grad} \underset{\sim}{x} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = \frac{\partial x_i}{\partial X_j} \underset{\sim}{e}_i \otimes \underset{\sim}{E}_j$$

Then $\det F = J > 0$ gives volume change: $dv = J\,dV$

Ex



$$\underset{\sim}{x} = \sum_{j=1}^{3} \frac{a_i}{A_j} X_j \underset{\sim}{e}_i$$

$$\Rightarrow F = \sum \frac{a_i}{A_i} \underset{\sim}{e}_i \otimes \underset{\sim}{E}_i = \operatorname{diag}\left(\frac{a_1}{A_1}, \frac{a_2}{A_2}, \frac{a_3}{A_3}\right)$$

④ Force balance (lin. momentum)

$$\frac{d}{dt}\int_{\Omega} \underbrace{\rho}_{\text{density}} \underbrace{\underset{\sim}{v}}_{\text{velocity}} \, dv = \int_{\Omega} \rho \underbrace{\underset{\sim}{b}}_{\text{body force (external)}} \, dv + \int_{\partial\Omega} \underbrace{\underset{\sim}{t}}_{\text{contact force}} \, dA$$

Cauchy: $\exists$ tensor T st $\underset{\sim}{t} = T \underset{\sim}{n}$ ← unit normal

Div Thm + Ω arb → $\boxed{\operatorname{div} T + \rho \underset{\sim}{b} = \rho \dot{\underset{\sim}{v}}}$

Angular momium → $\boxed{T^T = T}$ ☆ T Cauchy stress tensor

⑤ Hyperelastic material: $\exists$ strain-energy fn $W = W(F)$

such that elastic energy $= \int_B W \, dV$

⑥ Energy balance ⇝ $T = J^{-1} F \frac{\partial W}{\partial F}$ (compressible)

For incompressible $T = F \frac{\partial W}{\partial F} - p\mathbb{1}$ $J \equiv 1$, p Lagrange multiplier

7) Stretches

$d\underset{\sim}{X} = \underset{\sim}{M}\, dS$ $\longrightarrow$ $d\underset{\sim}{x} = \underset{\sim}{m}\, ds$

unit vec unit

$d\underset{\sim}{x} = F\, d\underset{\sim}{X}$ $\rightsquigarrow \Rightarrow \underset{\sim}{m}\, ds = F \underset{\sim}{M}\, dS$

$\Rightarrow |ds|^2 = (F\underset{\sim}{M}) \cdot (F\underset{\sim}{M}) |dS|^2$

norm

$\Rightarrow$ stretch $\dfrac{ds}{dS} = \sqrt{(F^T F \underset{\sim}{M}) \cdot \underset{\sim}{M}}$

characterises strain

($F^T F = \mathbb{1} \rightarrow$ material unstrained)

8) Polar Decomposition

: $\det F > 0 \Rightarrow \exists$ unique U, V symmetric, pos. definite, and orthogonal R ($R^T R = \mathbb{1}$, $\det R = 1$) such that

- $F = RU = VR$
- $F^T F = U^2 =: C$ $\leftarrow$ Right Cauchy Green tensor
- $F F^T = V^2 =: B$ $\leftarrow$ Left " " "

Can write $V = \sum_{i=1}^{3} \lambda_i \underset{\sim}{v}_i \otimes \underset{\sim}{v}_i$ $\{\lambda_i, \underset{\sim}{v}_i\}$ are eig vals, eig vec's of V

$\lambda_i \in \mathbb{R}$, $\lambda_i > 0$

(U has same λ_i)

- λ_i principal stretches
- $\underset{\sim}{v}_i$ directions of princ stretch

(9) Isotropic material (same response in any direction)

$$W(F) = W(V)$$

↖ rotations don't matter!

- Frame invariance (objectivity) $\Rightarrow$ W fn only of principal invariants of V.

OR, more convenient: express W ito invariants of $V^2 = B = FF^T$ (coeffs of charac. poly):

$$I_1 = \operatorname{tr} B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \operatorname{tr}(B^2)\right) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2$$

$$I_3 = \det B = \lambda_1^2\lambda_2^2\lambda_3^2$$

↳ so really $W = W(\lambda_1, \lambda_2, \lambda_3)$ (or sometimes $W(I_1, I_2, I_3)$)

• incompressible: $\lambda_3 = \dfrac{1}{\lambda_1\lambda_2}$, $I_3 \equiv 1$

(10) $$T = F\frac{\partial W}{\partial F} - p\mathbb{1} \quad \text{(incomp.)}$$

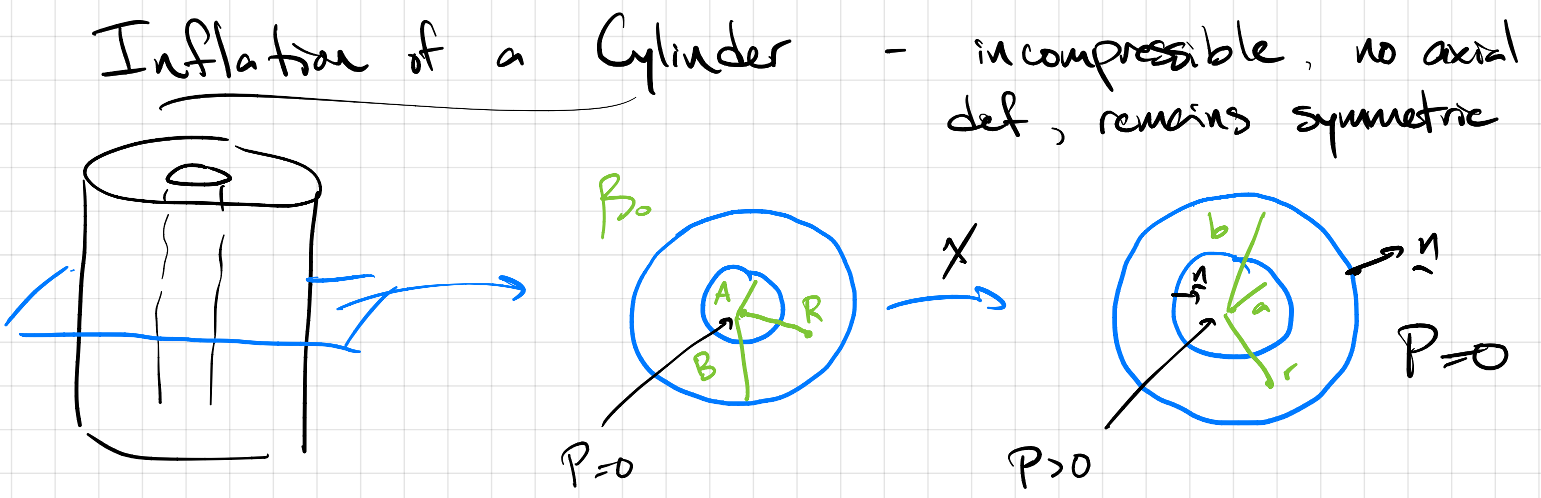
$\rightarrow$ $t_i = \lambda_i \dfrac{\partial W}{\partial \lambda_i} - p$ where $T = \sum_{i=1}^{3} t_i \, \underset{\sim}{v}_i \otimes \underset{\sim}{v}_i$

[in $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ basis, $F = V = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$

$\rightarrow$ $\dfrac{\partial W}{\partial F} = \operatorname{diag}\left(\dfrac{\partial W}{\partial \lambda_1}, \dfrac{\partial W}{\partial \lambda_2}, \dfrac{\partial W}{\partial \lambda_3}\right)$]

Inflation of a Cylinder

– incompressible, no axial def, remains symmetric



$$\underset{\sim}{X} = R\,\underset{\sim}{e}_r\,, \quad \underset{\sim}{x} = r(R)\,\underset{\sim}{e}_r\,, \quad \theta = \Theta$$

Then* $F = \dfrac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = r'(R)\,\underset{\sim}{e}_r \otimes \underset{\sim}{e}_r + \dfrac{r}{R}\,\underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta = \mathrm{diag}\Big(\underbrace{r'}_{\lambda_r}, \underbrace{\tfrac{r}{R}}_{\lambda_\theta}, 1\Big)$

- incompr: $\lambda_\theta = \dfrac{1}{\lambda_r} \Rightarrow r\,dr = R\,dR \Rightarrow r^2 - a^2 = R^2 - A^2$

  * deformation fully det'd once a known

- Force balance: $\mathrm{div}\,T = 0$

$$T = t_r(r)\,\underset{\sim}{e}_r \otimes \underset{\sim}{e}_r + t_\theta(r)\,\underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta$$

$$\left[\ \mathrm{div}\,T = \frac{\partial T}{\partial r}\cdot \underset{\sim}{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\cdot \underset{\sim}{e}_\theta\right.$$

$$= \frac{\partial}{\partial r}\left(t_r\,\underset{\sim}{e}_r \otimes \underset{\sim}{e}_r\right)\cdot \underset{\sim}{e}_r + \cancel{\frac{\partial}{\partial r}\left(t_\theta\,\underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta\right)\cdot \underset{\sim}{e}_r}\ 0$$

$$+ \frac{1}{r}\frac{\partial}{\partial \theta}\left(t_r\,\underset{\sim}{e}_r \otimes \underset{\sim}{e}_r + t_\theta\,\underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta\right)\cdot \underset{\sim}{e}_\theta \qquad \left[\ \frac{\partial \underset{\sim}{e}_r}{\partial \theta} = \underset{\sim}{e}_\theta,\quad \frac{\partial \underset{\sim}{e}_\theta}{\partial \theta} = -\underset{\sim}{e}_r\ \right]$$

$$= \left(\frac{\partial t_r}{\partial r} + \frac{t_r - t_\theta}{r}\right)\underset{\sim}{e}_r\ \Big]$$

$$\Rightarrow \boxed{\frac{dt_r}{dr} + \frac{t_r - t_\theta}{r} = 0}$$

Bdy cond : $\underbrace{T n}\cdot n = 0$ at $r=b$

$n = e_r$

$$\begin{cases} t_r = 0 & \text{at } r=b \\ t_r = -P & \text{at } r=a \quad (n = -e_r) \end{cases}$$

• Constit

$$\begin{cases} t_r = \lambda_r \dfrac{\partial W}{\partial \lambda_r} - p \\ t_\theta = \lambda_\theta \dfrac{\partial W}{\partial \lambda_\theta} - p \end{cases}$$

← unknown hydrostatic pressure

insert in FB → $$\frac{dt_r}{dr} = \frac{\lambda_\theta \frac{\partial W}{\partial \lambda_\theta} - \lambda_r \frac{\partial W}{\partial \lambda_r}}{r}$$

// $\int_a^b dr$ → $$\boxed{P = \int_a^b \frac{\lambda_\theta \frac{\partial W}{\partial \lambda_\theta} - \lambda_r \frac{\partial W}{\partial \lambda_r}}{r}\, dr}$$

given $A, B, W(\lambda_r, \lambda_\theta)$, and P

- this is an eqn for a

eg neo-Hookean (standard, "easiest non-lin")

$$W = \frac{\mu}{2}(I_1 - 3) = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \quad \mu = 3E$$

is shear modulus

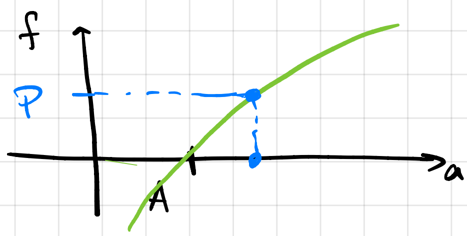
$$\Rightarrow \lambda_i \frac{\partial W}{\partial \lambda_i} = \mu \lambda_i^2$$

$$\Rightarrow P = \mu \int_a^b \frac{\lambda^2 - \frac{1}{\lambda^2}}{r}\, dr$$

$$\lambda := \lambda_\theta = \frac{r}{R} = \frac{\sqrt{a^2 + R^2 - A^2}}{R}$$

then $\lambda_r = \frac{1}{\lambda}$

$$\boxed{P = \mu \int_A^B \frac{\lambda(R)^2 - \lambda(R)^{-2}}{r(R)^2}\, R\, dR} = f(a)$$



* **Note:**

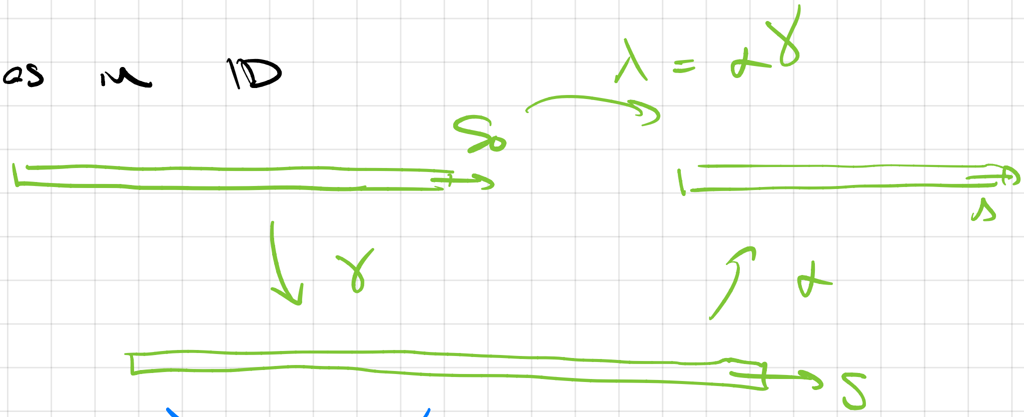
$$\underline{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = \frac{\partial \underset{\sim}{x}}{\partial R} \otimes \underset{\sim}{e}_r + \frac{1}{R} \frac{\partial \underset{\sim}{x}}{\partial \Theta} \otimes \underset{\sim}{e}_\theta$$

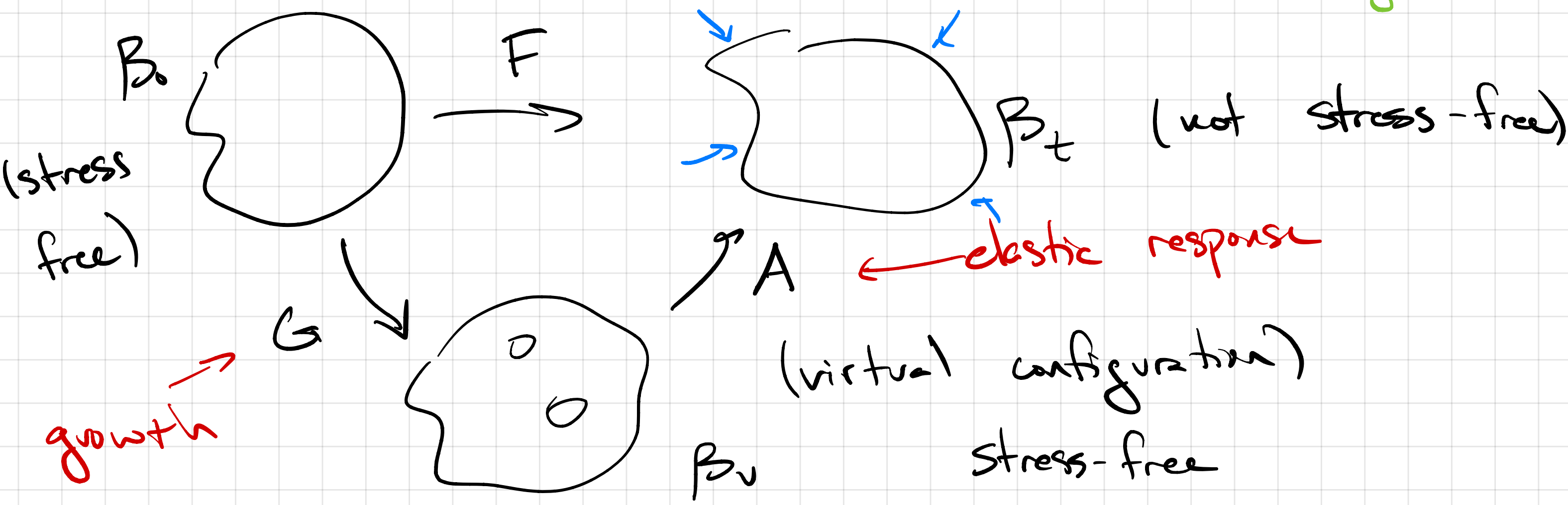
& $\underset{\sim}{x} = r(R)\, \underset{\sim}{e}_r$ $\qquad \dfrac{d\underset{\sim}{e}_r}{d\theta} = \underset{\sim}{e}_\theta$

$$\Rightarrow F = \frac{dr}{dR} \underset{\sim}{e}_r \otimes \underset{\sim}{e}_r + \frac{r}{R} \underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta$$

Morpoelasicity · a framework for growing elastic bodies

· same idea as in 1D

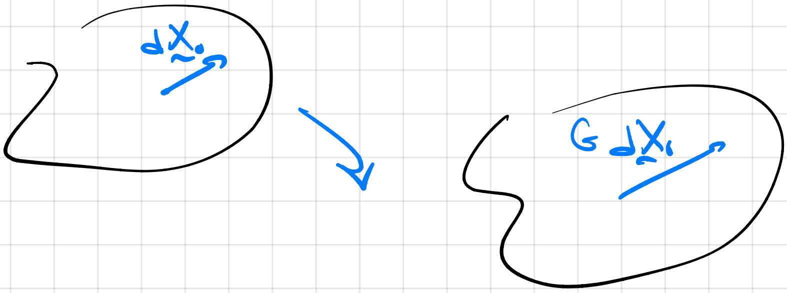




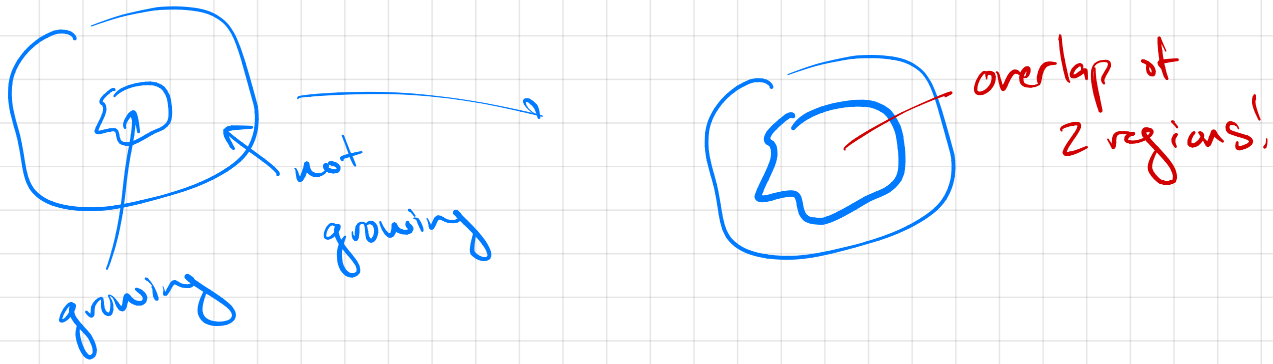
Multiplicative decomposition : $F = A\,G$

- G - growth tensor - describes local increase (or decrease) in mass
  - maps B_0 to virtual config B_v

· why a tensor? - growth can occur differently in different directions



- Growth can induce incompatibilities (eg holes, overlaps)



A - elastic tensor "restores compatibility"

⇝ stress only depends on elastic deformation:

$$\boxed{T = A \frac{\partial W}{\partial A} - p \mathbb{1}} \qquad \left(T = F \frac{\partial W}{\partial F} - p\mathbb{1} \right)$$

- all other eq'ns are same as before!

Types of Growth

- **Isotropic** - same in all directions

$$G = g\mathbb{1}$$

$$d\underset{\sim}{X}_0 \rightarrow g\,d\underset{\sim}{X}_0$$

- **Anisotropic** Not same in all directions

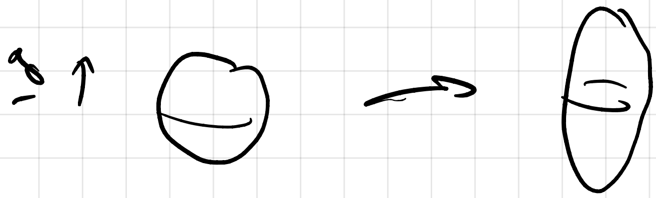
  - eg transversely isotropic - one growth direction $\underset{\sim}{g}$, $|\underset{\sim}{g}| = 1$

and let $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ be basis for direction orthog. to $\underset{\sim}{g}$

Then $$G = \gamma\, \underset{\sim}{g} \otimes \underset{\sim}{g} + \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2$$

Thus $$d\underset{\sim}{X}_0 = dX_0\, \underset{\sim}{g} \Rightarrow G\, d\underset{\sim}{X}_0 = \gamma\, d\underset{\sim}{X}_0$$

& if $$d\underset{\sim}{X}_0 = dX_0\, \underset{\sim}{e}_1 \Rightarrow G\, d\underset{\sim}{X}_0 = d\underset{\sim}{X}_0$$



Homogeneous same form of growth at all points

Heterogeneous growth is a function of position

Note Anisotropy and/or heterogeneity creates incompatibility $\Rightarrow$ the current config $\mathcal{B}_t$ may be stressed even if unloaded

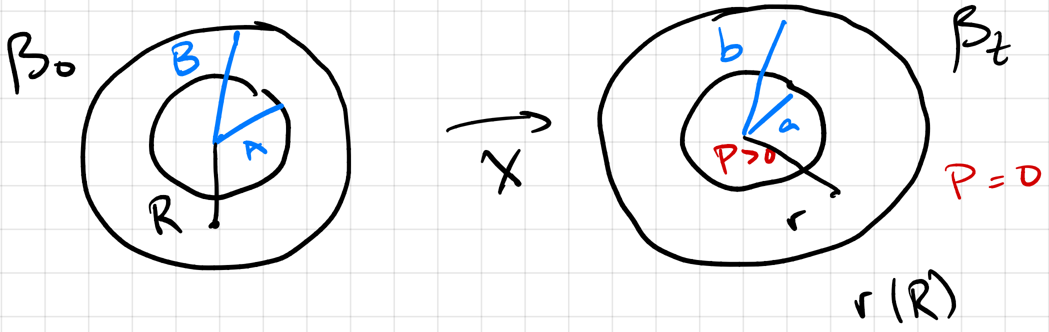
- this is called residual stress

very common and important in biological tissues

(arteries, skin, trees, ...)

Ex A Growing Cylinder

- As before, assume incompressible, no axial def, & symmetric def.

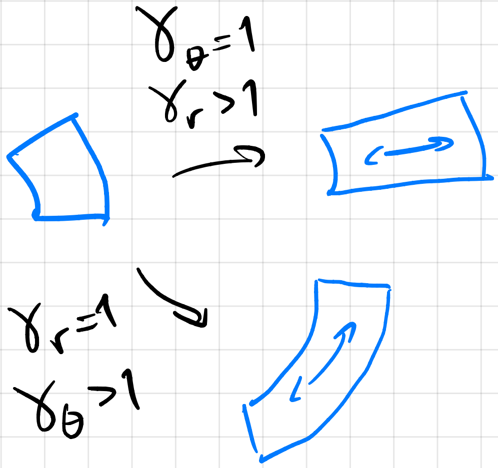


$$F = \operatorname{Grad} \chi = \operatorname{diag}\left(r'(R), \frac{r}{R}, 1\right)$$

We introduce $A = \operatorname{diag}(\alpha_r, \alpha_\theta, 1)$

& $G = \operatorname{diag}(\gamma_r, \gamma_\theta, 1)$

radial growth (γ_r), circumferential growth (γ_θ)



incompressible $\Rightarrow \det A = 1$

$$\Rightarrow \alpha_r = \frac{1}{\alpha_\theta} =: \frac{1}{\alpha}$$

$$F = AG \Rightarrow r'(R) = \alpha_r \gamma_r, \quad \frac{r}{R} = \alpha_\theta \gamma_\theta$$

$$\Rightarrow \frac{r}{\gamma_\theta R} = \frac{\gamma_r}{r'(R)} \Rightarrow r\,dr = \gamma_r \gamma_\theta R\,dR$$

Now integrate $\rightsquigarrow$ $$\tfrac{1}{2}(r^2-a^2) = \int_A^R \gamma_r(\hat{R})\,\gamma_\theta(\hat{R})\,\tilde{R}\,d\hat{R}$$

- given γ_r, γ_θ as fns of R, then the above defines def. $r(R)$ — but a is unknown!

<u>Force balance</u> $\quad \operatorname{div} T = 0$

$$T = \operatorname{diag}(t_r, t_\theta, 1) \rightarrow \frac{dt_r}{dr} + \frac{t_r - t_\theta}{r} = 0$$

<u>Bdry cond</u> $$\begin{cases} t_r(b) = 0 \\ t_r(a) = -P \end{cases}$$ imposed

<u>Constit</u> $$T = A\frac{\partial W}{\partial A} - p\mathbb{1}$$

unknown

in components: $$\begin{cases} t_r = \alpha_r \dfrac{\partial W}{\partial \alpha_r} - p \\ t_\theta = \alpha_\theta \dfrac{\partial W}{\partial \alpha_\theta} - p \end{cases}$$

$$\rightarrow \underbrace{P = \int_a^b \frac{\alpha_\theta \frac{\partial W}{\partial \alpha_\theta} - \alpha_r \frac{\partial W}{\partial \alpha_r}}{r}\,dr}$$

given $\{\gamma_r, \gamma_\theta, A, B, W, P\}$

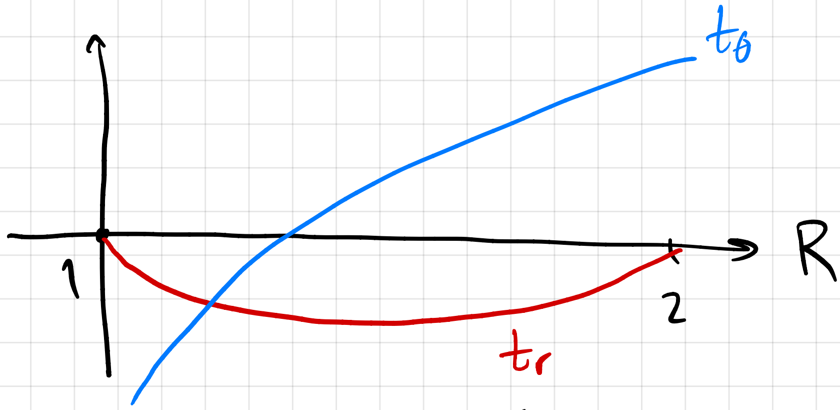
forms an eqn to find a

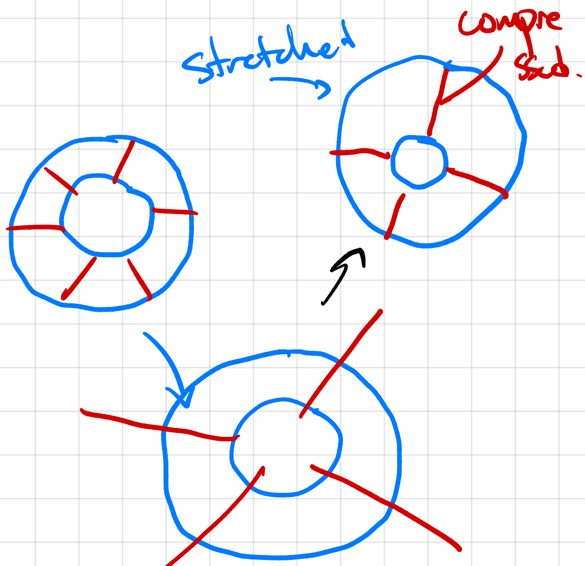
eg, neo-Hookean : $W = \frac{\mu}{2}\left(\underbrace{\alpha_r^2}_{=\frac{1}{\alpha^2}} + \underbrace{\alpha_\theta^2}_{=\alpha^2} + \underbrace{\alpha_z^2}_{=1} - 1\right)$

$\rightarrow \quad P = \mu \int_a^b \frac{\alpha^2 - \alpha^{-2}}{r}\, dr \quad , \quad \& \quad \alpha = \frac{r(R)}{\gamma_\theta R}$

$$\boxed{P = \mu \int_A^B \frac{\alpha(R)^2 - \alpha(R)^{-2}}{r(R)^2}\, \gamma_r \gamma_\theta R\, dR} \quad r\,dr = \gamma_r \gamma_\theta R\, dR$$

<u>Ex</u> $A = 1, \; B = 2, \; \mu = 1$





$P = 0, \; \gamma_r = 3, \; \gamma_\theta = 2 \quad \rightarrow \quad a \approx 1.64$

$\downarrow$ (compare: $\gamma_r = \gamma_\theta = 3 \Rightarrow a = 3,$

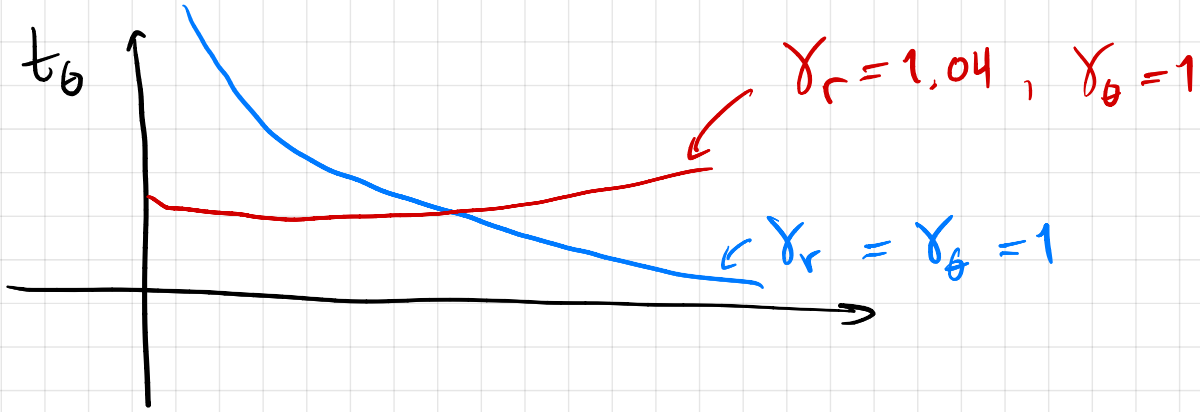
residual stress $\qquad b = 6$

t_θ - "hoop stress" $\qquad$ and $t_r = t_\theta = 0$)

$t_\theta(2) > 0$: circum. tension on outside

$t_\theta(1) < 0$: " " compression on inside

$t_r(R) < 0$: radial compression



$P = .1$

☆ residual stress can reduce stress gradients

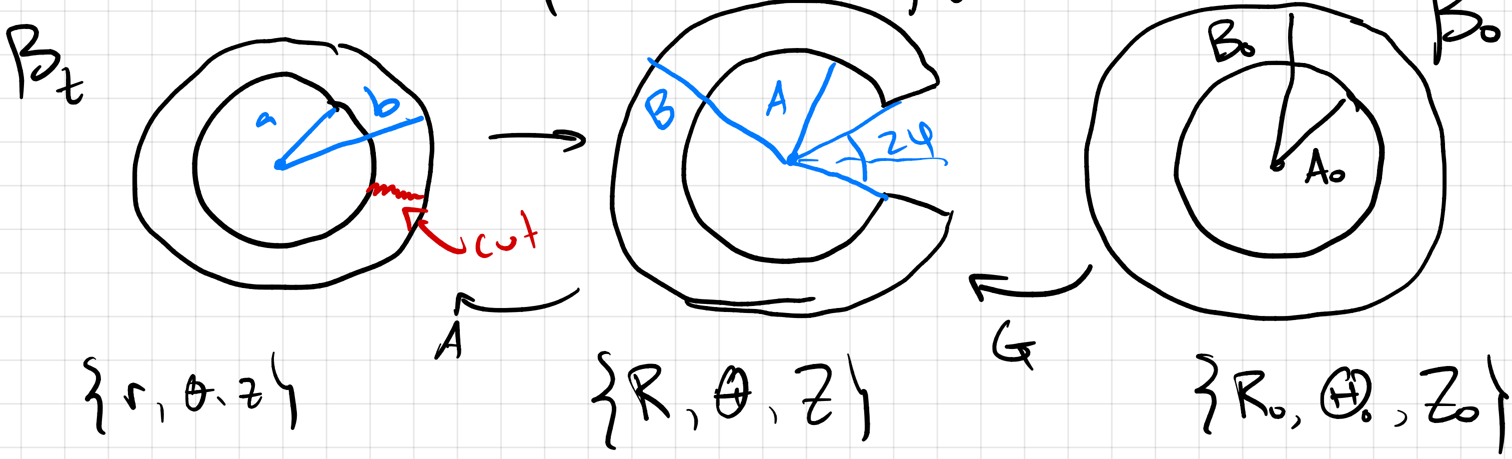
How measure residual stress?

In practice, we usually don't know G, and we have access to $\mathcal{B}_t$

- But whole framework requires knowing $\mathcal{B}_0$, $\mathcal{B}_r$!

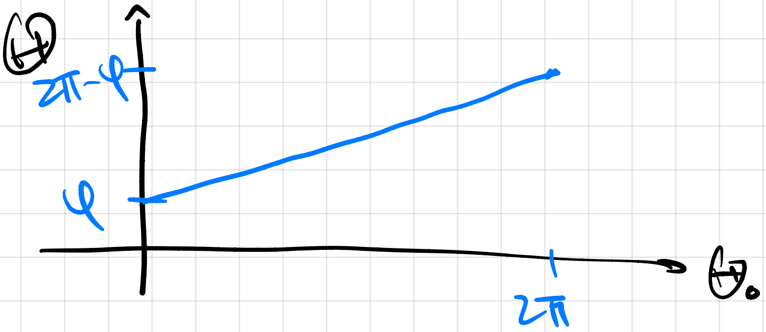
Possible resolution - determine G by relieving residual stress.

Ex. Opening angle test for residually stressed artery. Assume $\mathcal{B}_r$ (stress free)



Supp. for simplicity the map from $\mathcal{B}_0$ to $\mathcal{B}_r$ doesn't change radius or length:

$$R = R_0, \quad Z = Z_0, \quad \Theta = \varphi + \frac{2\pi - 2\varphi}{2\pi}\Theta_0$$



$$\underset{\sim}{R} = R\,\underset{\sim}{e}_R$$

$$= R_0\,\underset{\sim}{e}_R$$

Now, $\underset{\sim}{e}_R = \begin{pmatrix} \cos\Theta \\ \sin\Theta \\ 0 \end{pmatrix}$, $\underset{\sim}{e}_\Theta = \begin{pmatrix} -\sin\Theta \\ \cos\Theta \\ 0 \end{pmatrix}$

& $G = \operatorname{Grad}\underset{\sim}{R} = \dfrac{\partial}{\partial R_0}\underset{\sim}{R} \otimes \underset{\sim}{e}_R + \dfrac{1}{R_0}\dfrac{\partial}{\partial\Theta_0}\underset{\sim}{R} \otimes \underset{\sim}{e}_\Theta$

plug in $\underset{\sim}{R} = R_0\,\underset{\sim}{e}_R$

& use $\dfrac{\partial}{\partial\Theta_0}\underset{\sim}{e}_R = \underset{\sim}{e}_\Theta \cdot \Theta'(\Theta_0) = \left(1-\dfrac{\varphi}{\pi}\right)\underset{\sim}{e}_\Theta$

$$\Rightarrow G = \underset{\sim}{e}_R \otimes \underset{\sim}{e}_R + \left(1-\frac{\varphi}{\pi}\right)\underset{\sim}{e}_\Theta \otimes \underset{\sim}{e}_\Theta$$

ie $G = \operatorname{diag}\left(1,\ 1-\dfrac{\varphi}{\pi},\ 1\right)$

ie $\gamma_r = 1$, $\gamma_\theta = 1-\dfrac{\varphi}{\pi}$, $\gamma_z = 1$

so can compute stress, etc as before.