STRING THEORY I

Lecture 10

$\boxed{4}$ Interactions

4.1 Generalities

4.2 Vertex operators: introduction

4.3 Vertex operators: open string

4.4 The state vertex correspondence

open strings

closed string

4.5 3-point interactions

4.6 4-point tachyon amplitude

(4.7 Comments on the general picture)

4.4 State-vertex correspondence (continued)

Closed strings : analogous to the open string.

Recall a primary operator $\mathcal{A}(\tau,\sigma)$ of dimension $(h,\tilde{h})$ is an operator transforming under infinitesimal conformal transformations as

$$[L_m, \mathcal{A}(\sigma_I)] = \frac{1}{2} e^{2im\sigma_+} (-i\partial_+ + 2mh] \mathcal{A}(\sigma_I)$$

$$[\tilde{L}_m, \mathcal{A}(\sigma_\pm)] = \frac{1}{2} e^{2im\sigma_-} (-i\partial_- + 2m\tilde{h}] \mathcal{A}(\sigma_I)$$

(total derivatives if $h=\tilde{h}=1$)

The vertex operator

$$:e^{ik\cdot X(\sigma_I)}:$$

is primary with $h = \tilde{h} = \frac{e^2}{8} K\cdot K$ so

$$V_T(K;\sigma_I) = :e^{ik\cdot X(\sigma_I)}: \qquad e^2 K\cdot K = 8$$

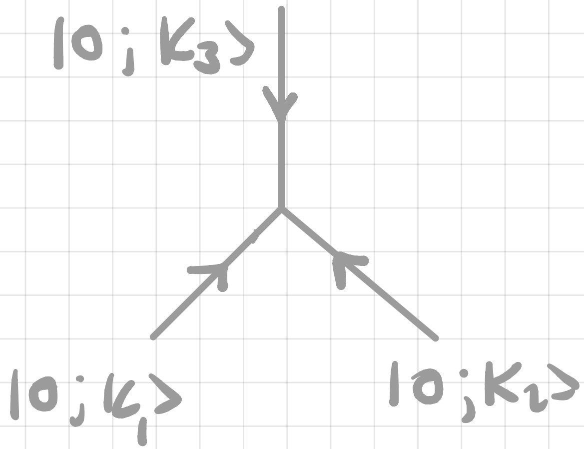
etc.

The map Vertex operators $\leftrightarrow$ states is given by

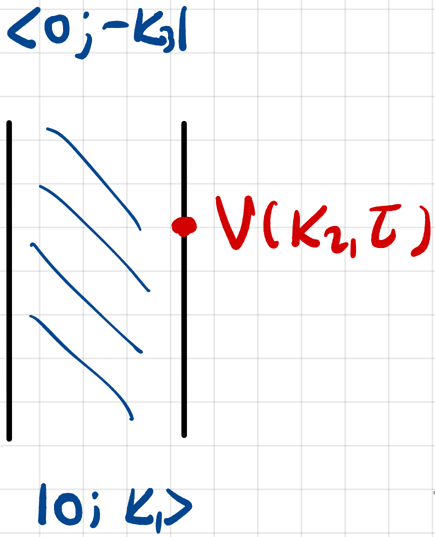
$$|\psi\rangle = \lim_{t\to\infty} (z\bar{z})^{-1} V_\psi(it,\sigma)|0;0\rangle, \quad z = e^{2(t-i\sigma)}$$
$$\bar{z} = e^{2(t+i\sigma)}$$

4.5 Three point interactions

three point open string interaction eg tachyon absorving a tachyon at $\sigma=0$, τ



$\longleftrightarrow$



$$\mathcal{A}_3^{oo}(k_1,k_2,k_3) = g_o \int_{-\infty}^{\infty} d\tau \, \langle 0;-k_3 | V_T(\tau;k_2) | 0;k_1\rangle \Big/ \mathrm{Vol(conf.)}$$

open string coupling constant

infinite volume of the gauge group

divide out by a divergent "volume" due to the residual gauge symmetry (τ-translations of V_T leaving past & future states invariant)

$$L_0 = \frac{\ell^2}{2}p^2 + N \to 1 + 0$$

$$A_3^{\infty}(k_1,k_2,k_3) = g_0 \int d\tau\, \underbrace{\langle 0;-k_3|\, e^{i\tau L_0}}_{\langle 0;-k_3|\, e^{i\tau}}\, V_T(0;k_2)\, \underbrace{e^{-i\tau L_0}\, |0;k_1\rangle}_{e^{-i\tau}|0;k_1\rangle} \Big/ \mathrm{Vol(conf.)}$$

$$= g_0 \int d\tau\, \langle 0;-k_3|\, \underbrace{V_T(0;k_2)}_{\vdots}\, |0;k_1\rangle \Big/ \mathrm{Vol(conf.)}$$

$$\underbrace{e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-m}}{m}}}_{1+\text{creation operators}} : e^{ik\cdot x} : \underbrace{e^{-k\sum_{n=1}^{\infty}\frac{\alpha_n}{n}}}_{1+\text{annihilation operators}}$$

$$= g_0 \int d\tau\, \langle 0;-k_3|\, \underbrace{: e^{ik_2\cdot x} :\, |0;k_1\rangle} \Big/ \mathrm{Vol(conf.)}$$

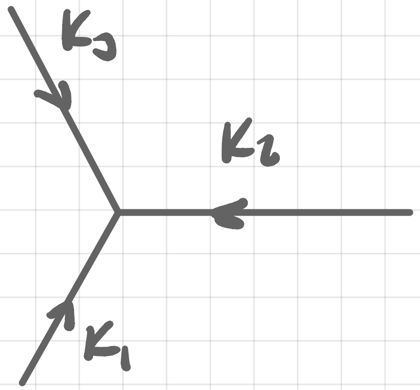
$$\mathcal{A}_3^{\infty}(k_1,k_2,k_3) = g_0 \int d\tau \, \langle 0; -k_3 | 0; k_1+k_2 \rangle \Big/ \mathrm{Vol(conf.)}$$

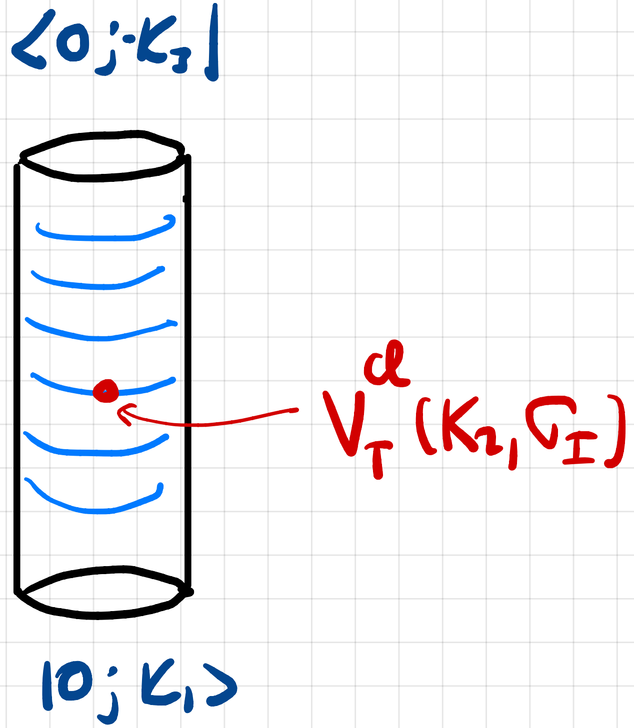
$$= g_0 \, \delta(k_1+k_2+k_3) \left[\int_{-\infty}^{\infty} d\tau \Big/ \mathrm{Vol(conf.)} \right]$$

alternatively: gauge fix at $\tau = 0$!

$$\mathcal{A}_3^{\infty}(k_1,k_2,k_3) = g_0 \, \delta(k_1+k_2+k_3)$$

tree point tree level closed string diagrams (tachyon absorbing a tachyon)





$$A_3^{cl}(k_1, k_2, k_3) = g_{cl} \int d^2\sigma_I \, \langle 0; -k_3| \, V_T^{cl}(k_2, \sigma_I) \, |0; k_1\rangle \Big/ \mathrm{Vol(conf.)}$$

$$V_T^{cl}(K,\sigma_\pm) = e^{2i\sigma_- L_0 + 2i\sigma_+ \tilde{L}_0}\, V_T^{cl}(K,0)\, e^{-2i\sigma_- L_0 - 2i\sigma_+ \tilde{L}_0}$$

$H = 2(L_0 + \tilde{L}_0)$

$L_0 = \frac{\ell^2 p^2}{8} + N$

$L_0 |0;K_1\rangle = |0;K_1\rangle$

$$A_3^{cl}(K_1,K_2,K_3)$$

$$= g_{cl} \int d^2\sigma_\pm \langle 0;-K_3|\, e^{2i\sigma_- L_0 + 2i\sigma_+ \tilde{L}_0}\, V_T^{cl}(K_2,0)\, \underbrace{e^{-2i\sigma_- L_0 - 2i\sigma_+ \tilde{L}_0}}_{e^{-2i\sigma_- - 2i\sigma_+}} |0;K_1\rangle \Big/ \mathrm{Vol(conf.)}$$

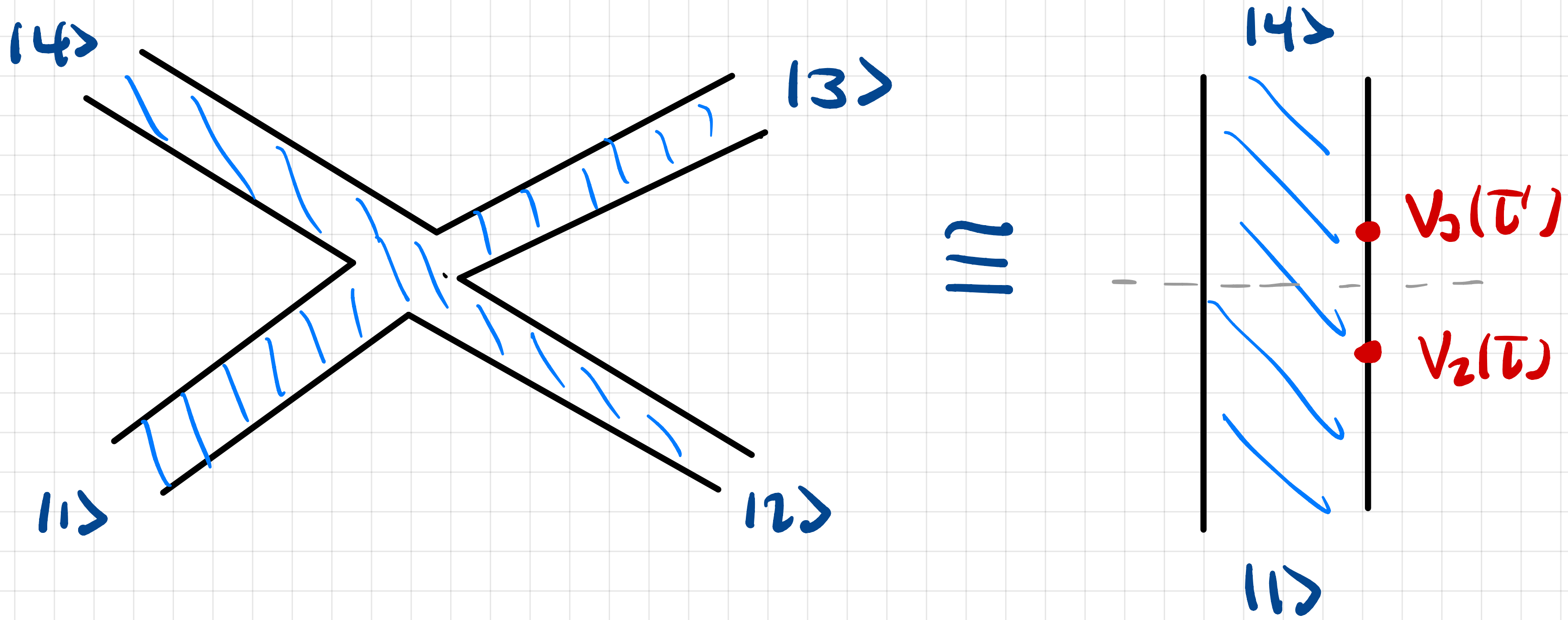
$$= g_{cl} \int d^2\sigma_\pm \langle 0;-K_3|\, \underbrace{V_T^{cl}(K_2,0)}_{(1+\text{creation})\, e^{iK_2\cdot x}\, (1+\text{annihilation})} |0;K_1\rangle \Big/ \mathrm{Vol(conf.)}$$

$$A_3^{cl}(k_1,k_2,k_3) = g_{cl} \int d^2\sigma_\pm \langle 0; -k_3 | 0; k_1+k_2 \rangle \Big/ \text{Vol(conf.)}$$

$$= g_{cl}\, \delta(k_1+k_2+k_3) \int d^2\sigma_\pm \Big/ \text{Vol(conf.)}$$

$$\underline{A_3^{cl} = g_{cl}\, \delta(k_1+k_2+k_3)}$$

$\boxed{4.6}$ 4-point tachyon amplitude (open string)



$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{\tau'>\tau} d\tau'\, d\tau \, \langle 0;-k_4| \, V_3(\tau') V_2(\tau) \, |0;k_1\rangle \Big/ \mathrm{Vol(conf)}$$

$$= g_0^2 \int_{\tau'>0} d\tau'\, d\tau \, \langle 0;-k_4| \, e^{i\tau' L_0} V_3(0)\, e^{-i\tau' L_0} \, V_2(\tau)\, |0;k_1\rangle \Big/ \mathrm{Vol(conf)}$$

Use the residual gauge freedom to fix $\tau' = 0$

$$\Phi_4(K_1, K_2, K_3, K_4) = g_0^2 \int_{-\infty}^{0} d\tau \; \langle 0; -K_4 | V_3(0) V_2(\tau) | 0; K_1 \rangle$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \langle 0, -K_4 | V_3(0) e^{i\tau L_0} V_2(0) \underbrace{e^{-i\tau L_0} |0; K_1\rangle}_{e^{-i\tau} |0; K_1\rangle}$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \; \langle 0; -K_4 | V_3(0) e^{i\tau(L_0 - 1)} V_2(0) | 0; K_1 \rangle$$

$$= g_0^2 \langle 0; -K_4 | V_3(0) \underbrace{\left(\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0 - 1)} \right)}_{\text{propagator}} V_2(0) | 0; K_1 \rangle$$

What do we do with $\int_{-\infty}^{0} d\bar{\tau}\, e^{i\bar{\tau}(L_0-1)}$?

$$\int_{-\infty}^{0} d\bar{\tau}\, \underline{e}^{i\bar{\tau}(L_0-1)} \underline{|\cdots\rangle} \stackrel{?}{=} \left(\frac{-i + \text{oscillatory modes}}{L_0 - 1} \right) |\cdots\rangle \qquad ??$$

Consider instead

$$\int_{-\infty}^{0} d\bar{\tau}\, e^{i\bar{\tau}(L_0-1-i\epsilon)}$$

converges

[Analogy in QFT:

$$\int_{-\infty}^{0} d\bar{\tau}\, e^{i\bar{\tau}(p^2+m^2-i\epsilon)} = -i \frac{1}{p^2+m^2-i\epsilon}$$

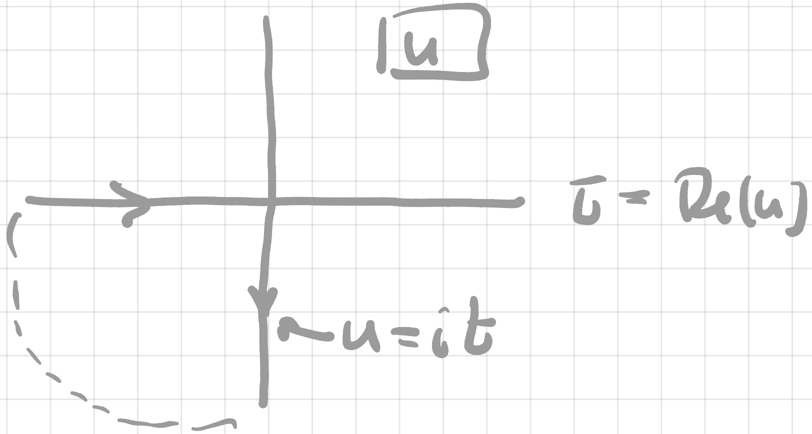
shift exponent to avoid poles due to on-shell particles]

Now "rotate" the contour: $\tau = -it$

no need of $-it$

$$\int_{-\infty}^{0} d\tau\, e^{i\tau(l_0-1-i\epsilon)} |\dots\rangle \longrightarrow \int_{-\infty}^{0} dt\, e^{t(l_0-1)} |\dots\rangle$$

close the contour



and consider

$$\oint du\, e^{iu(l_0-1-i\epsilon)} = 0 \quad \text{(no poles inside contour)}$$

breaks up into the two integrals $\int_{-\infty}^{0} d\tau \dots + i\int_{0}^{-\infty} du\, e^{\dots}$

$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_0^2 \langle 0;-k_4|V_3(0)\left(\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1)}\right)V_2(0)|0;k_1\rangle$$

$$= g_0^2 \langle 0;-k_4|V_3(0)\left(\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1-i\epsilon)}\right)V_2(0)|0;k_1\rangle$$

$$\Big\downarrow \text{ rotate contour } \tau = -it$$

$$= g_0^2 \langle 0;-k_4|V_3(0)\left(\int_{-\infty}^{0} dt\, e^{t(L_0-1)}\right)V_2(0)|0;k_1\rangle$$

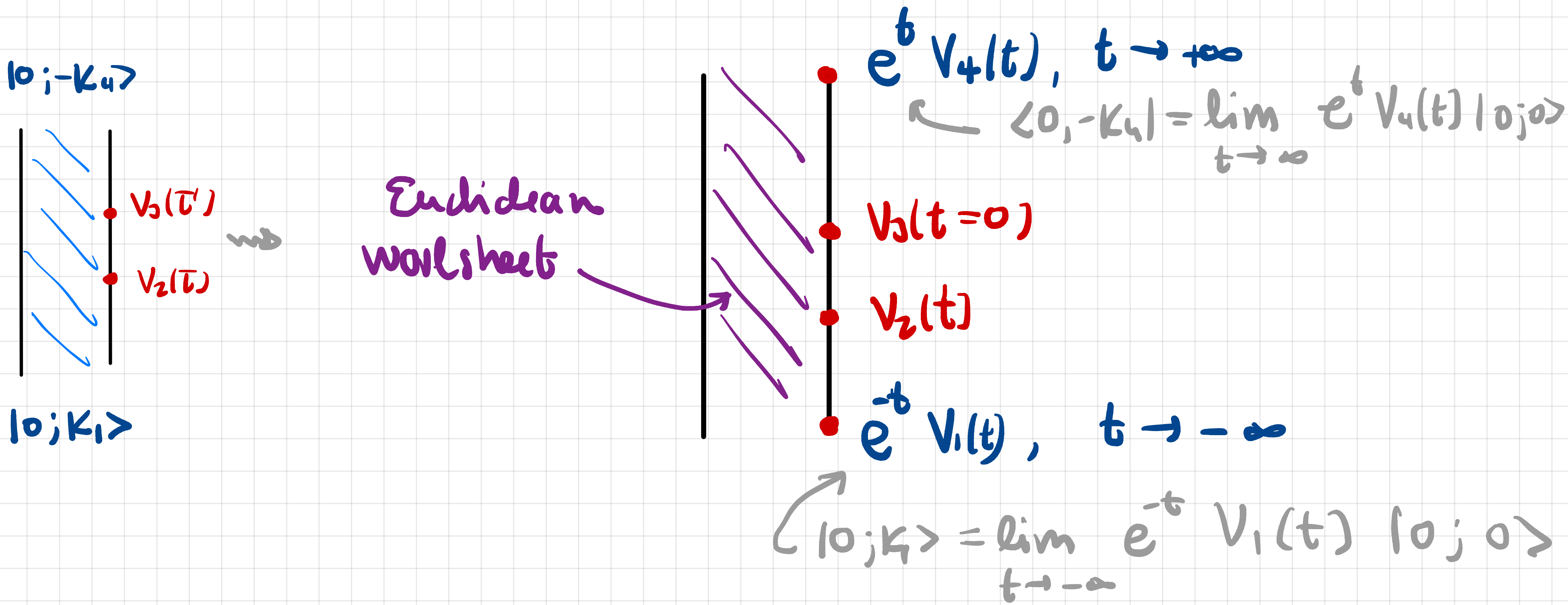
$$= g_0^2 \int_{-\infty}^{0} dt\, \langle 0;-k_4|V_3(0)\, e^{tL_0}\, V_2(0)\, e^{-tL_0}|0;k_1\rangle$$

$$= g_0^2 \int_{-\infty}^{0} dt\, \langle 0;-k_4|V_3(0)\, V_2(it)\,|0;k_1\rangle$$

all operators are now in <u>Euclidean</u> worldsheet time.

The amplitude now has an interpretation in Euclidean worldsheet.

$$A_4(k_1, k_2, k_3, k_4) = g_o^2 \int_{-\infty}^{0} dt \, \langle 0; -k_4 | V_3(0) \, V_2(t) | 0; k_1 \rangle$$



Consider now a Euclidean conformal map

$$z = e^{t+i\sigma} \quad , \quad \bar{z} = e^{t-i\sigma}$$

Weyl trans

metric: so $dz\,d\bar{z} = e^{2t}\,(dt^2 + d\sigma^2) \longrightarrow dt^2 + d\sigma^2$

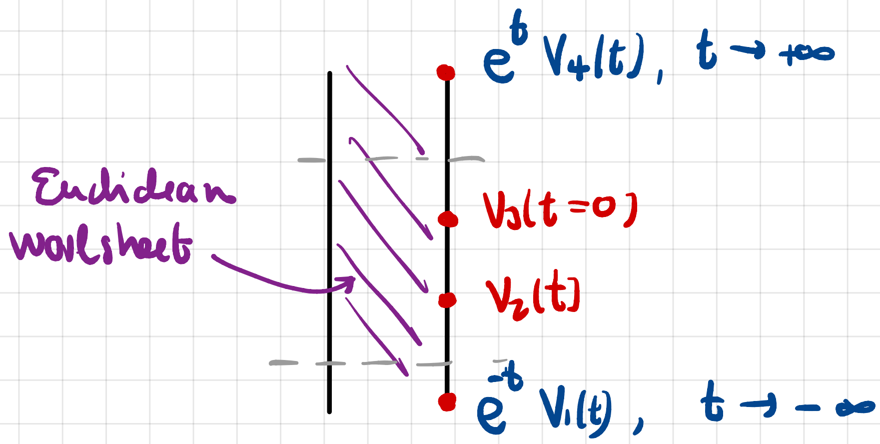
$|z|^2 = e^{2t}$ ie $\dfrac{dz\,d\bar{z}}{|z|^2} = dt^2 + d\sigma^2$

This is the metric on the upper half-plane (UHP)

$\mathrm{Im}\, z = e^{t} \sin\sigma$

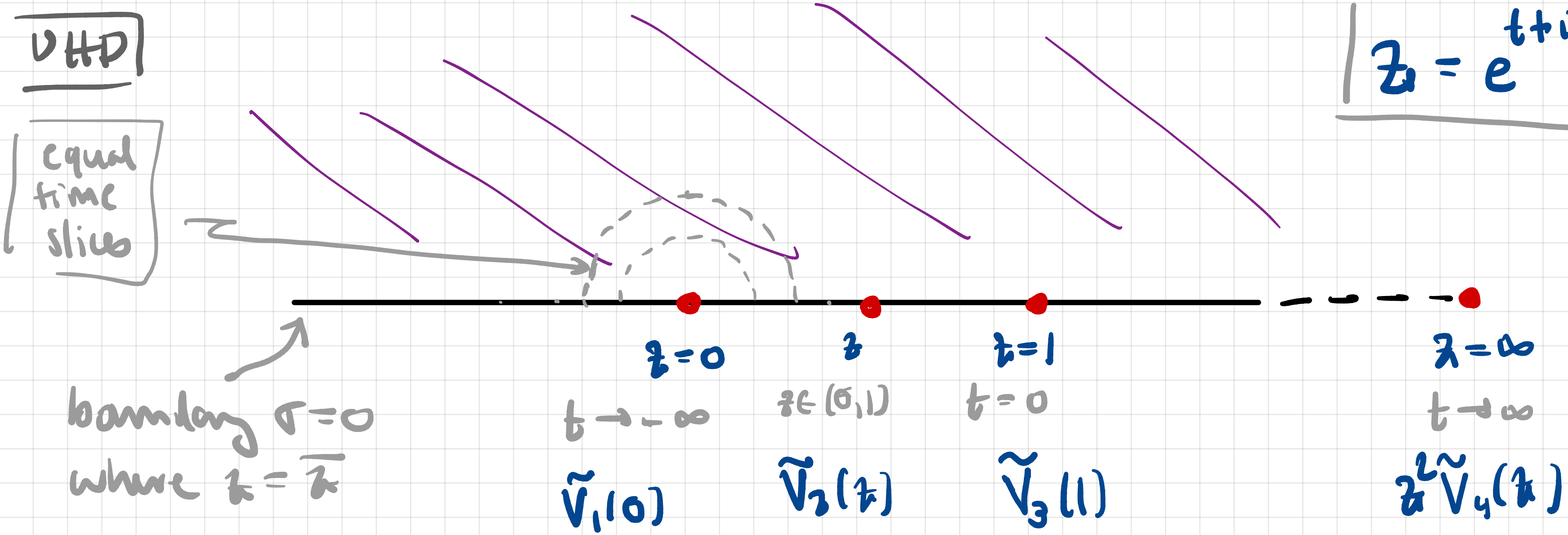
but $0 \le \sigma \le \pi \Rightarrow \sin\sigma \ge 0 \Longrightarrow \mathrm{Im}\, z \ge 0$

The strip



is mapped to the UHP with four marked points

$$\boxed{z = e^{t+i\sigma}}$$



$\tilde{V}(z)$ vertex operator change

Modification of the vertex operators due to the conformal transformation:

Recall that a vertex operator $V(\tau)$ transforms as

$$V(\tau) \longrightarrow \tilde{V}(\tilde{\tau}) = \left(\frac{d\tau}{d\tilde{\tau}}\right)^1 V(\tau) \qquad (h=1)$$

Then

$$\tilde{V}(\underbrace{z=\bar{z}}_{\sigma=0}) = \frac{dt}{dz} V(t) = z^{-1} V(t)$$

$$z = e^{t+i\sigma} \Longrightarrow \frac{dz}{dt} = z$$

ie $$z\,\tilde{V}(z=\bar{z}) = V(t)$$

Thus for the operator V_i we have:

- incoming state $|0;k_1\rangle = \lim_{t\to-\infty} e^{-t} V_1(t)|0;0\rangle$

$$\lim_{t\to-\infty} e^{-t} V_1(t) = \lim_{z\to 0} \tilde{V}_1(z)$$

- $V_2(t)$: $V_2(t)\,dt = z\,\tilde{V}_2(z)\,\frac{1}{z}\,dz = \tilde{V}_2(z)\,dz$

- $V_3(t=0)$: $V_3(t=0) = \tilde{V}_3(1)$

- outgoing state $\langle 0;-k_4| = \lim_{t\to\infty} e^{t} V_4(t)$

$$\lim_{t\to\infty} e^{t} V_4(t) = \lim_{z\to\infty} z^2 \tilde{V}_4(z)$$

End of lecture 10

Next:
- more on conformal transformation
- gauge fixing procedure
- Comments on the general picture