Problem Sheet 3

1. Find the communicating classes of the Markov chains with the following transition matrices on the state space $\{1,2,3,4,5\}$, and in each case determine which classes are closed:

$$\text{(i)} \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{(ii)} \quad \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}.$$

   If X is a chain with the transition matrix in (ii), find the distribution of X_1 when X_0 has the uniform distribution on $\{1,2,3,4,5\}$, and find $P(X_2 = 3 | X_0 = 1)$.

2. N black balls and N white balls are distributed between two urns, numbered 1 and 2, so that each urn contains N balls. At each step, one ball is chosen at random from each urn and the two chosen balls are exchanged. Let X_n be the number of white balls in urn 1 after n steps. Find the transition matrix for the Markov chain X.

3. A die is "fixed" so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probablity $1/5$. If the first score is 6, what is the probability that the nth score is 6 and what is the probability that the nth score is 1? [*Hint: you can simplify things by selecting an appropriate state-space; do you really need a* 6*-state chain to answer the question?*]

4. Let $X_n, n \geq 1$ be i.i.d. taking value 1 with probability p and -1 with probability $1-p$, where $p \in (0,1)$. In each of the following cases, decide whether $Y_n, n \geq 1$ is a Markov chain. If so, find its transition probabilities.

   (a) $Y_n = X_n$.
   (b) $Y_n = S_n$ where $S_n = X_1 + X_2 + \cdots + X_n$.
   (c) $Y_n = M_n$ where $M_n = \max(0, S_1, S_2, \dots, S_n)$.
   (d) $Y_n = M_n - S_n$.
   (e) $Y_n = X_n X_{n+1}$.

5. Let C be a communicating class of a Markov chain. Prove the following statements:

   (a) Either all states in C are recurrent, or all are transient. (So we may refer to the whole class as transient or recurrent.) [*Hint: use the criterion for recurrence of a state i in terms of $\sum p_{ii}^{(n)}$ to show that if i is recurrent and $i \leftrightarrow j$ then also j is recurrent.*]
   (b) If C is recurrent then C is closed.
   (c) If C is finite and closed, then C is recurrent.

6. A gambler has £8 and wants to increase it to £10 in a hurry. He can repeatedly stake money on the toss of a fair coin; when the coin comes down tails, he loses his stake, and when the coin comes down heads, he wins an amount equal to his stake, and his stake is returned.

   He decides to use a strategy in which he stakes all his money if he has less than £5, and otherwise stakes just enough to increase his capital to £10 if he wins. For example, he will stake £2 on the first coin toss, and afterwards will have either £6 or £10.

(a) Let £X_n be his capital after the nth coin toss. Show how to describe the sequence $X_0, X_1, X_2, \ldots$ as a Markov chain.

(b) Find the expected number of coin tosses until he either reaches £10 or loses all his money.

(c) Show that he reaches £10 with probability 4/5.

(d) Show that the probability that he wins the first coin toss, given that he eventually reaches £10, is 5/8. Extend this to describe the distribution of the whole sequence $X_0, X_1, X_2, \ldots$ conditional on the event that he reaches £10.

(e) In a similar way, let $X_n, n \geq 0$ be a Markov chain on $\mathbb{N}$ with $p_{i,i+1} = p = 1 - p_{i,i-1}$ for $i \geq 1$, and $p_{0,0} = 1$. Let $p > 1/2$ so that the process has an upward bias. Start at $X_0 = j > 0$. In lectures we showed that the probability of absorption at 0 is $\left(\frac{1-p}{p}\right)^j$. Describe the distribution of $(X_n, n \geq 0)$ conditional on the event of being absorbed at 0.

7. A Markov chain with state space $\{0, 1, 2, \ldots\}$ is called a "birth-and-death chain" if the only non-zero transitions from state i are to states $i-1$ and $i+1$.

Consider a general birth-and-death chain and write $p_i = p_{i,i+1}$ and $q_i = p_{i,i-1} = 1 - p_i$. Assume that p_i and q_i are positive for all $i \geq 1$.

Let h_i be the probability of reaching 0 starting from i, and write $u_i = h_{i-1} - h_i$.

(a) Show that $p_i h_i + q_i h_i = h_i = p_i h_{i+1} + q_i h_{i-1}$, and hence that $u_{i+1} = \dfrac{q_i}{p_i} u_i$.

(b) Define $\gamma_i = \dfrac{q_1}{p_1}\dfrac{q_2}{p_2}\cdots\dfrac{q_{i-1}}{p_{i-1}}$.
Write u_i in terms of γ_i and u_1, and then h_i in terms of $\gamma_1, \ldots, \gamma_i$ and u_1.

(c) The equations for $h_1, h_2, \ldots$ may have multiple solutions. Which solution gives the true hitting probabilities? Hence find the value of u_1, and deduce that the chain is transient if and only if $\sum_{i=1}^{\infty} \gamma_i$ is finite.

(d) Consider the case where
$$p_i = \left(\frac{i+1}{i}\right)^2 q_i.$$
Show that if $X_0 = 1$, then $\mathbb{P}(X_n \geq 1 \text{ for all } n \geq 1) = 6/\pi^2$.

Additional problems:

8. Suppose P is an irreducible transition matrix, with period d. Consider the transition matrix P^k. In terms of d and k, how many communicating classes does P^k have, and what is the period of each state?

9. Consider a random walk on a cycle of size M; that is, a Markov chain with state space $\{0, 1, \ldots, M-1\}$ and transition probabilities
$$p_{ij} = \begin{cases} 1/2 & \text{if } j \equiv i+1 \mod M \\ 1/2 & \text{if } j \equiv i-1 \mod M \\ 0 & \text{otherwise} \end{cases}.$$
The walk starts at 0. What is the distribution of the last site to be reached by the chain?

10. Continuing question 4, let $Y_n = |S_n|$. Does this give a Markov chain?