STRING THEORY I

Lecture 11

4 Interactions

4.1 Generalities

4.2 Vertex operators: introduction

4.3 Vertex operators: open string

4.4 The state vertex correspondence

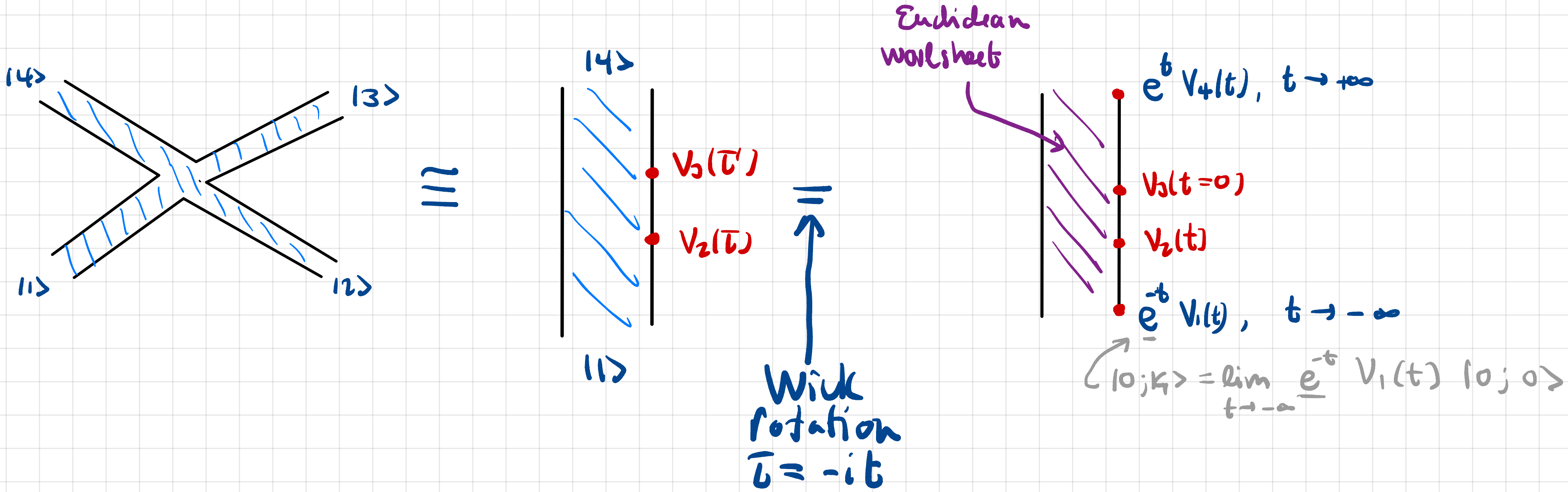
4.5 3-point interactions

4.6 4-point tachyon amplitude

4.7 Comments on the general picture

4.6 4-point tachyon amplitude (continued)

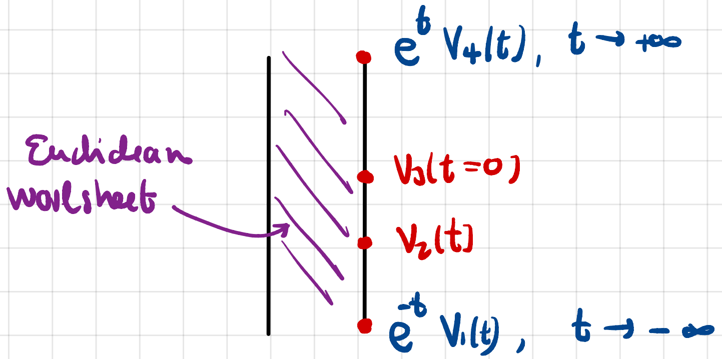
Last lecture: summary



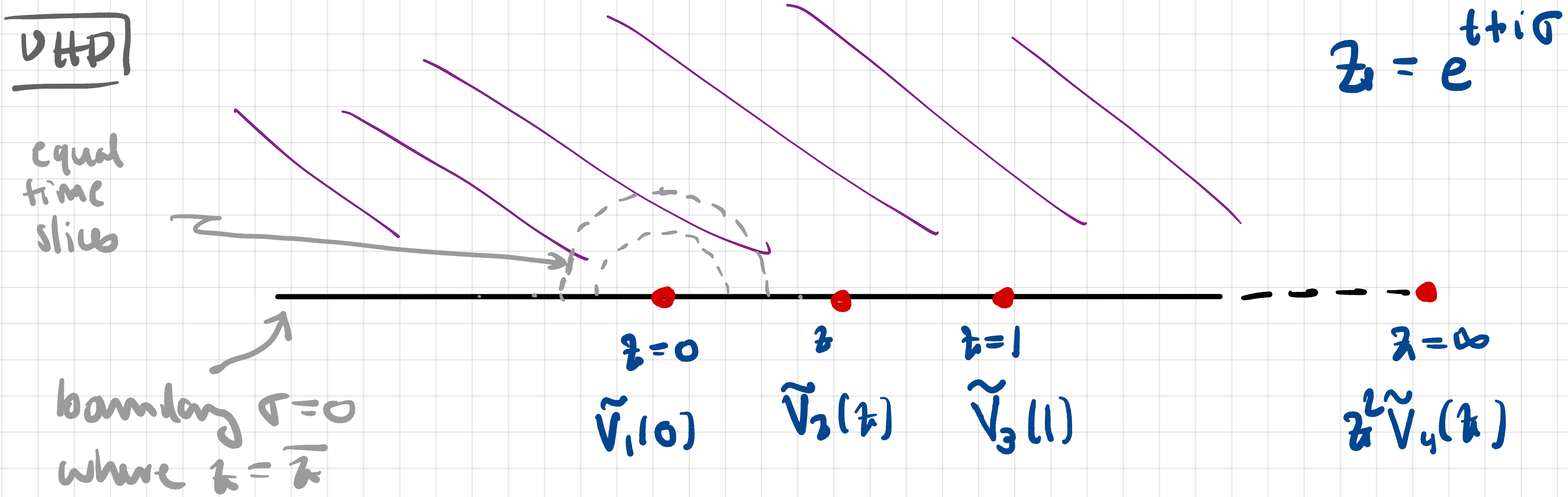
$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} dt \, \langle 0;-k_4| V_3(0) V_2(t) |0;k_1\rangle$$

Moreover: introduced a new coordinate $z = e^{t+i\sigma}$ which maps the Euclidean WS to the UHP

The strip



is mapped to the UHP with four marked points



For the operator $V_i \longrightarrow \tilde{V}_i$ we have: $z\,\tilde{V}(z) = V(t)$

$z=0$	z	$z=1$	$z=\infty$
$\tilde{V}_1(0)$	$\tilde{V}_2(z)$	$\tilde{V}_3(1)$	$z^2\tilde{V}_4(z)$

- incoming state $\quad |0;k_1\rangle = \lim_{t\to-\infty} e^{-t}\, V_1(t)\,|0;0\rangle$

$$\lim_{t\to-\infty} e^{-t}\, V_1(t) = \lim_{z\to 0} \tilde{V}_1(z)$$

- $V_2(t)$: $\quad V_2(t)\,dt = z\,\tilde{V}_2(z)\,\frac{1}{z}\,dz = \tilde{V}_2(z)\,dz$

- $V_3(t=0)$: $\quad V_3(t=0) = \tilde{V}_3(1)$

- outgoing state $\quad \langle 0;-k_4| = \langle 0;0|\, \lim_{t\to\infty} e^{t}\, V_4(t)$

$$\lim_{t\to\infty} e^{t}\, V_4(t) = \lim_{z\to\infty} z^2\, \tilde{V}_4(z)$$

$\rightarrow$ We can now express the amplitudes on the UHP

Let's try to discuss conformal transformations and the gauge fixing procedure in more generality

The group of conformal transformation of the upper-half plane is $PSL(2,\mathbb{R})$ — L_{-1}, L_0, L_1

Mobius group

2x2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$z \longmapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

maps UHP → UHP

This is a three dimensional group of residual gauge symmetries.

One can find a transformation which maps any three points z_1, z_3, z_4

$$z_1 \longrightarrow 0 \qquad z_3 \longrightarrow 1 \qquad z_4 \longrightarrow \infty$$

Indeed, for any four points z_1, z_3, z_4

set $$z_{ij} = z_i - z_j \quad (i \neq j)$$

Then $$z \longmapsto \frac{z_{34}}{z_{13}} \frac{z_1 - z}{z - z_4}$$

maps $z_1 \to 0$, $z_3 \to 1$ and $z_4 \to \infty$.

One can use this to gauge fix the tree point amplitude for example.

Of particular interest for us is the fact that the group of conformal transformations $PSL(2,\mathbb{R})$ of the UHP preserves the cyclic ordering of any four points on the boundary.

Consider a fourth point z_2, with $z_1 < z_2 < z_3 < z_4$ (all four points on the boundary)

Then: $\quad z_2 \longmapsto \dfrac{z_{12} z_{34}}{z_{13} z_{24}} \in (0,1)$ conformal cross ratio

map above

so maps $[z_1, z_2, z_3, z_4] \longrightarrow [0, \frac{z_{12} z_{34}}{z_{13} z_{24}}, 1, \infty]$

Fixing three points z_1, z_3 & z_4 at 0, 1 and ∞, then, the fourth point $0 < z_2 < 1$

Preservation of the cyclic ordering of 4 points on the boundary implies a cyclic symmetry of the 4 point amplitude.

This is what we are integrating over in the four point amplitude!

$$A_4(k_1,k_2,k_3,k_4) = g_o^2 \int_{-\infty}^{0} dt \, \langle 0; -k_4 | V_3(0)\, V_2(t) | 0; k_1 \rangle$$

$$\int_{-\infty}^{0} dt \longrightarrow \int_{0}^{1} dz$$

We say that $(0,1)$ is the moduli space of conformal structures on the UHP with four marked points.

We will comment briefly on this notion of moduli space later

Fadeev-Popov gauge fixing:

We could have written the four point amplitude as

$$g_o^2 \, \overbrace{\int dz_1\, dz_2\, dz_3\, dz_4}^{\text{overcounts configurations}} \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \rangle_{UHP} \Big/ Vol(SL(2;\mathbb{R}))$$

(very nice: invariant under conf transf)

and then use the Fadeev-Popov procedure to fix the gauge.

Imposing: $\quad z_1 = z_1^0 \qquad z_3 = z_3^0 \qquad z_4 = z_4^0$

$$g_o^2 \int dz_1\, dz_2\, dz_3\, dz_4\, \underbrace{\delta(z_1 - z_1^0)\, \delta(z_3 - z_3^0)\, \delta(z_4 - z_4^0)}_{\text{gauge choice}} \underbrace{\left| \operatorname{Det} \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right|}_{\text{Fadeev-Popov determinant}}$$

$$\times \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \rangle_{UHP}$$

$$\left| \operatorname{Det} \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right| = \text{Jacobian of the transformation from } z_1, z_2, z_3 \text{ to } \lambda_1, \lambda_0, \lambda_{-1}$$

$$= (z_4 - z_3)(z_3 - z_1)(z_4 - z_1)$$

where $\delta z = \underset{L_{-1}}{\lambda_{-1}} + \underset{L_0}{\lambda_0} z + \underset{L_{+1}}{\lambda_1} z^2$

i. Infinitesimal Möbius transformation

ii. expand around identity matrix: $a = d = 1$, $c = b = 0$ $\Rightarrow$

$$\lambda_{-1} = \delta b, \quad \lambda_0 = 2\delta a, \quad \lambda_1 = -\delta c$$

$$\text{Jac} = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_3 & z_4 \\ z_1^2 & z_3^2 & z_4^2 \end{vmatrix}$$

rows $\rightarrow \partial z^i / \partial \lambda$

$$= z_{43}\, z_{31}\, z_{41}$$

$d\mu$ = measure on UHP

$$g_o^2 \int d z_1\, d z_2\, d z_3\, d z_4\, \delta(z_1 - z_1^0)\,\delta(z_3 - z_3^0)\,\delta(z_4 - z_4^0)\, |(z_4 - z_3)(z_3 - z_1)(z_4 - z_1)|$$

$$\times \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \hat{V}_1(z_1) \rangle_{UHP}$$

choose $z_1^0 = 0$, $z_3^0 = 1$, $z_4^0 = \Lambda \to \infty$

$$= \lim_{\Lambda \to \infty} g_o^2 \int_0^1 d z_2\, (\Lambda - 1)\Lambda \langle \tilde{V}_4(\Lambda)\, \tilde{V}_3(1)\, \tilde{V}_2(z_2)\, \tilde{V}_1(0) \rangle_{UHP}$$

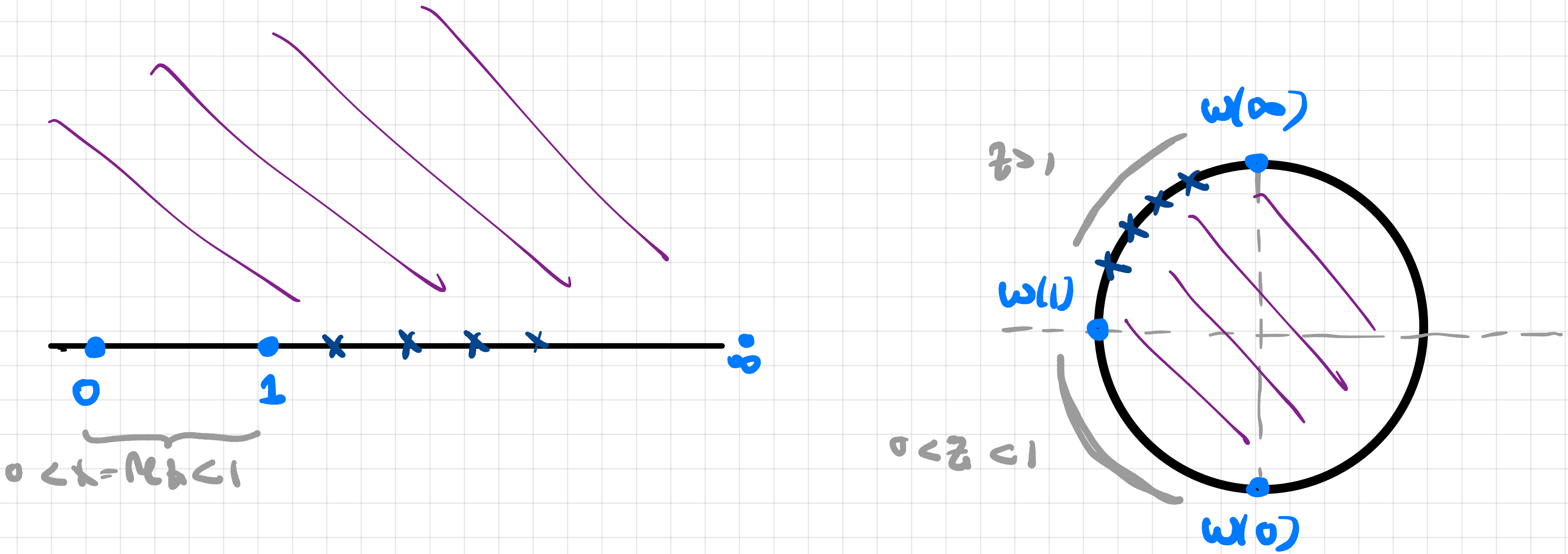
$$= g_o^2 \int d z \langle 4 | \tilde{V}_3(1)\, \hat{V}_2(z) | 1 \rangle$$

PS 3 This is the Veneziano amplitude.

cyclic symmetry: an elegant way to make the cyclic symmetry of amplitudes manifest is by introducing the map

$$z \longmapsto w = \frac{iz-1}{z-i}$$

which maps UHP $\longrightarrow$ unit disk



• boundary $z = \bar{z}$ $\longrightarrow$ boundary of unit disk $|w|^2 = 1$

14.7 Comments on the general picture

In string perturbation theory we are interested in the amplitude for the scattering of asymptotic in and out states (the S-matrix)

We have discussed a number of ideas and tools for computing amplitudes.

Wrap up this chapter on interactions with a number of comments on the lessons learned and on the general picture for scattering amplitudes

To study string amplitudes we use

physical states $\longleftrightarrow$ vertex correspondence

$|\psi\rangle \in \mathcal{H}_{phys} \longleftrightarrow V_\psi$ primary operator of conformal

$V: \mathcal{H}_{phys} \to \mathcal{H}_{phys}$ $\longrightarrow$ weight $\begin{cases} h = 1 & \text{open strings} \\ h = \tilde{h} = 1 & \text{closed strings} \end{cases}$

$\mathcal{H}_{null} \to \mathcal{H}_{null}$

V_ψ represents emission/absorption of a physical string state $|\psi\rangle$ from a point on the world sheet

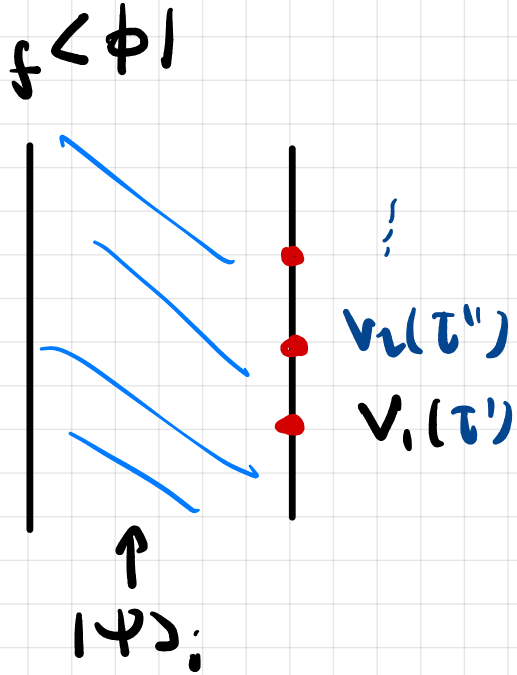
and incoming/outgoing states are represented by

$$|\psi\rangle = \lim_{z \to 0} z^{-1} V_\psi(z) |0;0\rangle$$

(action of V_ψ on zero momentum vacuum state in the infinite Euclidean past)

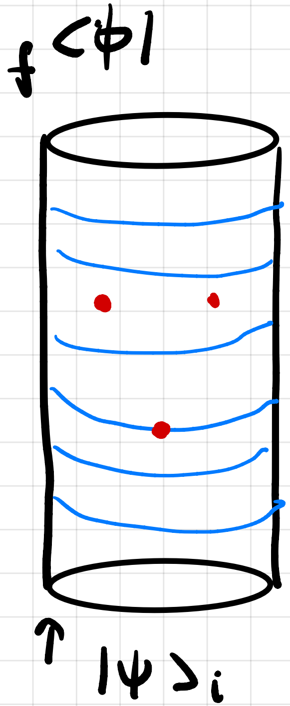
$$\langle\phi| = \lim_{z \to \infty} z \langle 0;0| V_\psi(z)$$

open strings (tree level) amplitudes



Vertex operators inserted on the boundary of the world sheet

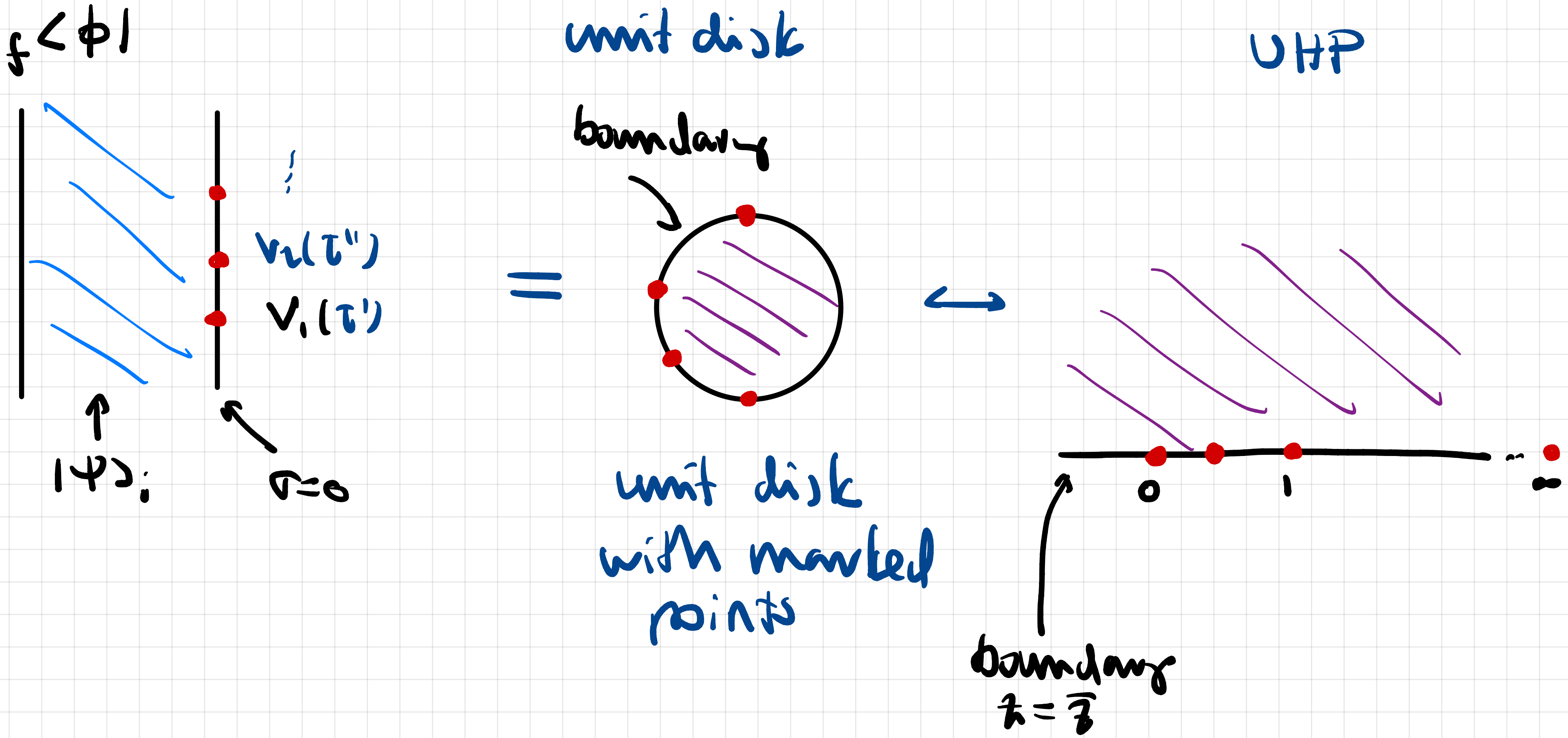
closed strings (tree level) amplitudes



Vertex operators inserted in the interior of the world sheet

Moreover, by a Wick rotation together with wise coordinate changes we map the Lorentzian world sheet into Euclidean world sheet and the amplitudes have now an interpretation on this Euclidean world sheet.

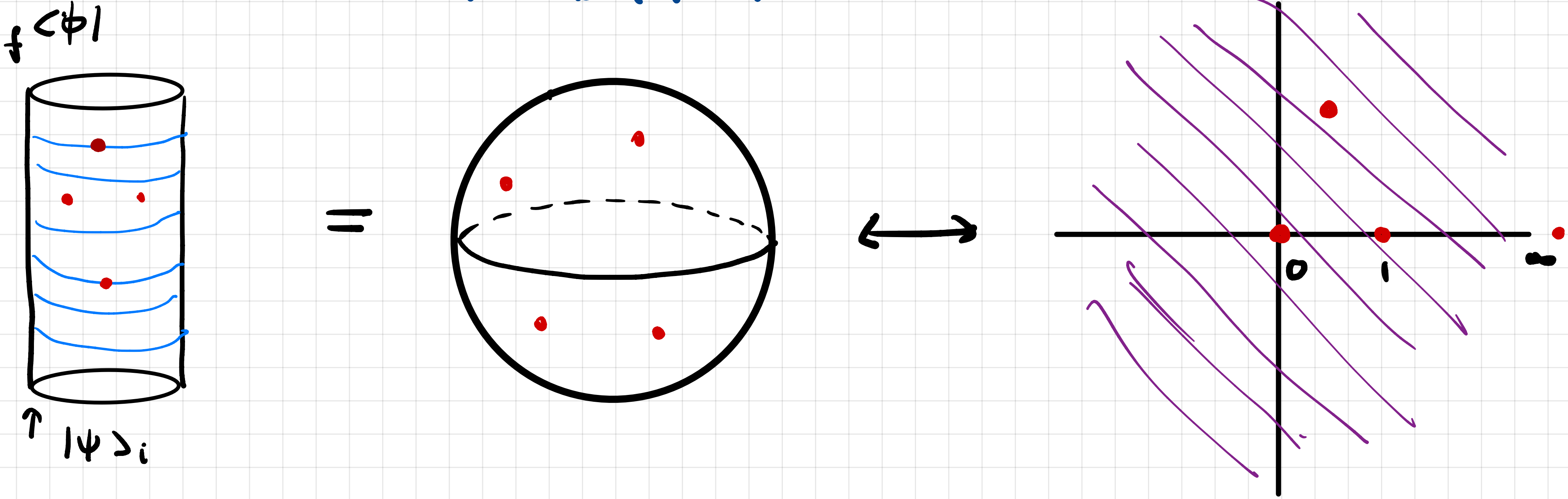
open strings
(tree level)



closed strings
(tree level)

sphere with marked points

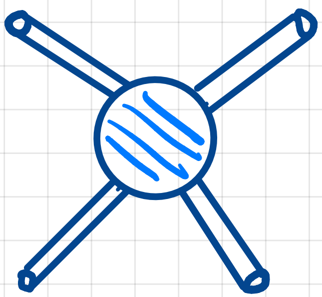
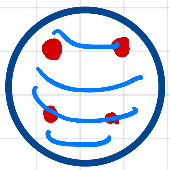
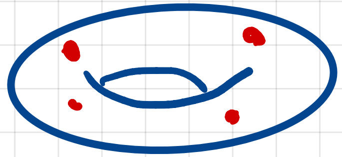
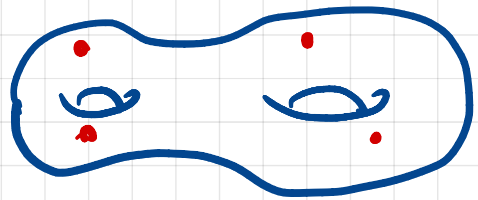
complex plane $\cup \{\infty\}$

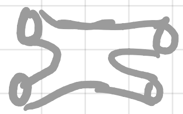


The string perturbation series is a <u>genus</u> expansion

Closed string

tree — 1-loop — 2-loops

 $=g_c^2$  $+\,g_c^4$  $+\,g_c^6$  $+\cdots$

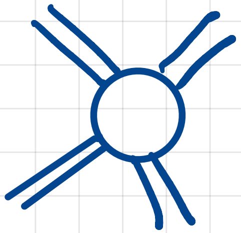
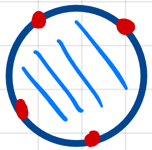
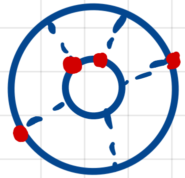
 

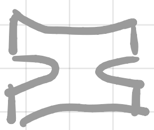
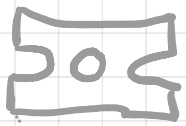
without boundaries

sum over all topologies of Riemann surfaces

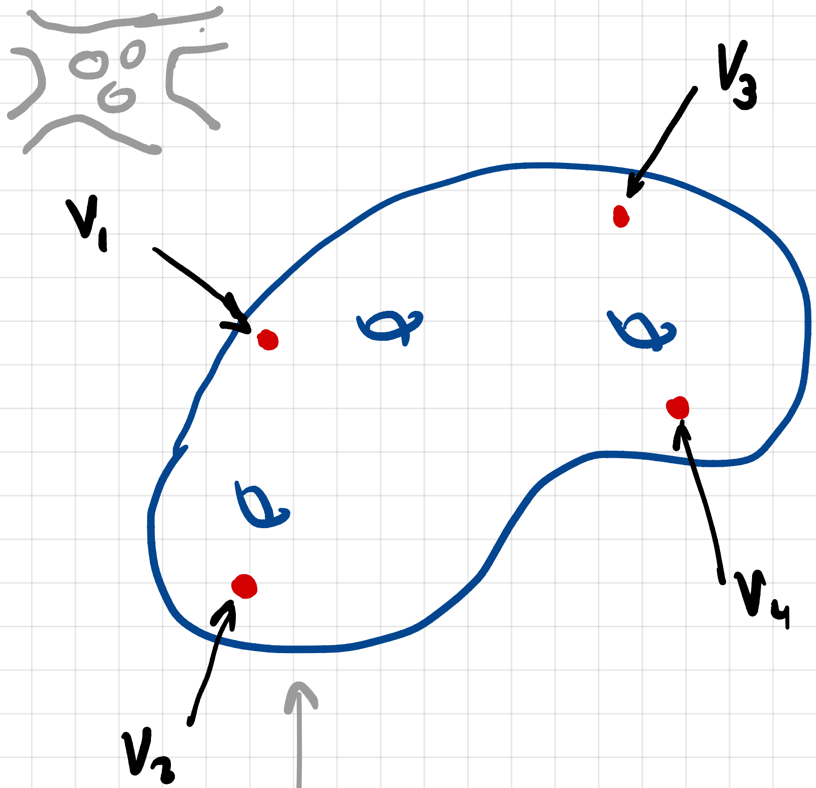
Open string

with boundaries

 $=g_o^2$  $+\,g_o^4$  $+\cdots$

At each order in perturbation level: one diagram



genus g Riemann surfaces (C-surfaces)

$$\rightsquigarrow A_g(\psi_1, \dots, \psi_4) = \int_{\mathcal{M}_{g,4}} \underbrace{[d\mu]}_{\text{measure on } \mathcal{M}} \langle V_1 V_2 V_3 V_4 \rangle_{\Sigma_{g,4}(\mu)}$$

parametrizes Weyl equivalent classes of metrics ie C-structures

Moduli space of Riemann surfaces with marked points

- M_Σ & $[d\mu]$ $\rightarrow$ complicated

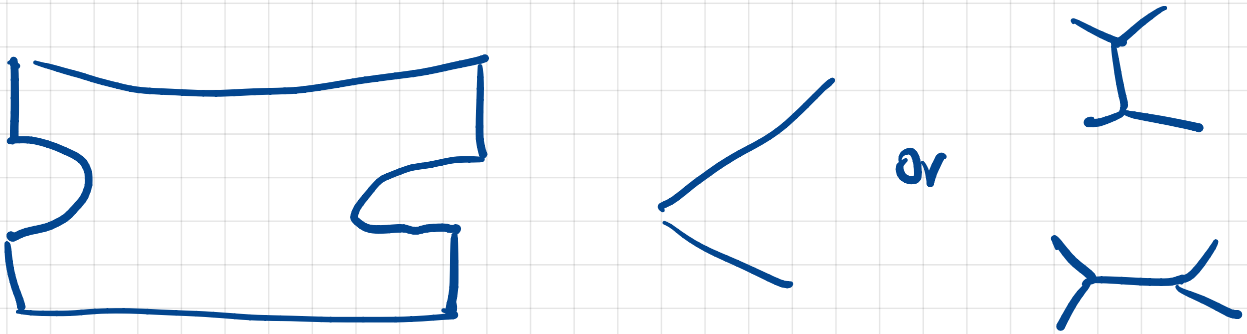
however low genus isn't so bad (we did tree level examples)
(1-loop amplitude calculations are rather interesting)

Remarks:

1. One diagram per order in perturbation theory

2. Degeneration limits look like many Feynman diagrams

eg



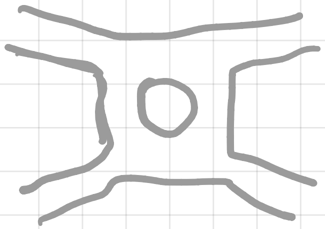
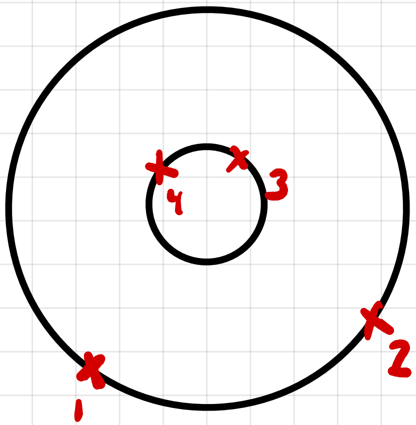
etc

3. Generalization of DHS duality

$\boxed{4}$ An interesting generalization of DHS duality is the open/closed duality : (to be discussed in Lecture 12)

Consider the following 1-loop open string amplitude : the annulus amplitude

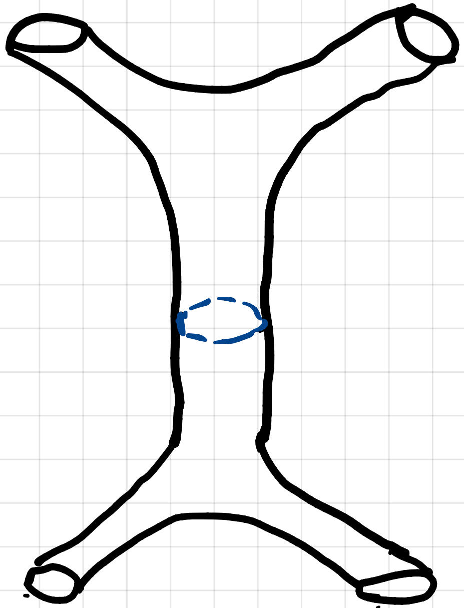
say for 2 incoming string states $|1\rangle, |2\rangle$ and two outgoing string states $\langle 3|, \langle 4|$

 $\rightsquigarrow$  $\sim g_o^4$

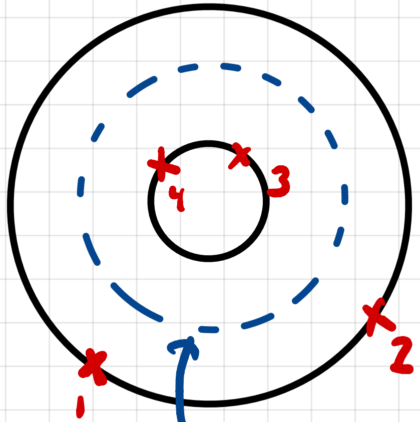
open string in a loop

But topologically this is the same as a cylinder so instead we can reinterpret this one loop open string amplitude as a tree level closed string amplitude

C. Beem magical amplitude



$\sim g_c^2$



a single geometry with two interpretations

closed string state

Consistency : $g_c = g_o^2$

$g_c \sim$ grav. coupling
$g_o \sim U(1)$ gauge coupling.

5 Final remark: UV behavior (see D Tong's lecture notes)

- UV-finiteness holds for all loop diagrams up to 2 loops.
  ↳ 2-loop finite theory of gravity interacting with matter in higher dims

However so far there is no proof for all loops ....

Next: strings in background fields

Let's try to discuss conformal transformations and the gauge fixing procedure in more generality.

Recall: (lecture 4)

A conformal transformation of a (Riemannian or Lorentzian) manifold Σ is a diffeomorphism $\xi \longmapsto \tilde{\xi}(\xi)$ that preserves the metric up to rescaling ie

$$\gamma_{..}(\xi) \longmapsto \tilde{\gamma}_{..}(\tilde{\xi}) = e^{2\Lambda(\tilde{\xi})}\gamma_{..}(\tilde{\xi})$$

(a special case is an isometry for which $\Lambda = 0$)

The infinitesimal conformal transformations can be described explicitly.

In fact, a conformal transformation

$$\delta\eta^{\alpha\beta} = \underbrace{(\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha)}_{\text{diffeo}} - \Lambda\eta^{\alpha\beta}$$

combination of : diffeo + Weyl

where $\delta\sigma^+ = \xi^+(\sigma^+)$ $\delta\sigma^- = \xi^-(\sigma^-)$, $\Lambda = \partial_-\xi^+ + \partial_+\xi^-$

leaves η invariant.

Then we have found the classical bosonic string theory is invariant under a larger group of symmetries