STRING THEORY I

Lecture 12

5 Strings in background fields

5.1 Background field expansion and the Weyl anomaly

We have identified various massless fields in the bosonic string spectrum. In particular, we identified a graviton in the closed string spectrum.

We expect then that a theory of space-time gravity should emerge, so we suppose that spacetime should be allowed a non trivial metric (or indeed non trivial topology).

Indeed we expect a $D=26$ dim theory of gravity emerging with a Hilbert-Einstein action

The action for a string moving in a spacetime with metric $G_{\mu\nu}(X)$ is

$$S_P[\gamma, X] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \gamma^{ab}\partial_a X^\mu \partial_b X^\nu \underbrace{G_{\mu\nu}(X)}_{\text{target space metric}}$$

where, so far, we have only considered a flat spacetime metric ($G_{\mu\nu} = \eta_{\mu\nu}$)

Classically this is Weyl invariant so taking $\gamma_{ab} = e^{2\phi(\sigma)}\eta_{ab}$

$$S_P[X] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_a X^\mu \partial^a X^\nu G_{\mu\nu}(X)$$

NON-LINEAR σ-MODEL

NLσM

describes an interacting 2dim QFT with couplings encoded in the target space metric $G_{\mu\nu}(X)$

complicated! compare $G_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$ free field theory

In this chapter we discuss how a $D=26$ dim gravitational theory emerges: we will do this from the effective field theory point of view.

KEY: require that the quantum theory is Weyl invariant

First however use this action to try to make sense of the graviton states in the string spectrum.

(we will generalize the action later to include the other massless states)

Consider a spacetime manifold with metric

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \gamma_{\mu\nu}$$

small perturbation of flat space

We can compute amplitudes in this background treating the fluctuation as a perturbative parameter:

$$S_P[G] = S_P[\eta] - \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_a X^\mu \partial^a X^\nu \gamma_{\mu\nu}$$

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Then

Fourier transform to momentum space

$$\langle A_1 \cdots A_n \rangle = \underbrace{\langle A_1 \cdots A_n \rangle_0}_{\text{amplitude in a flat background}} - \frac{1}{4\pi\alpha'}\langle A_1 \cdots A_n \int d^2z \underbrace{\gamma_{\mu\nu}\, \partial_a X^\mu \partial^a X^\nu e^{ik\cdot X}}_{\text{graviton vertex operator}} \rangle + \cdots$$

(as long as $\gamma_{\mu\nu}$ satisfies the appropriate conditions)

Returning to the action $S_P[G]$:

We would like to require that the 2 dim QFT on the world sheet (ie NLσ-M) to be Weyl invariant at the quantum level.

This implies, in particular, that the theory is conformally invariant.

why? essential for the consistency of the theory (construction of states, vertex operators and amplitudes based on having a CFT on the WS

This requirement places restrictions on target space fields.

However the NLσ-M is not so easy to analyze.

To analyse the quantum NLσM we use the covariant background field expansion, which is a perturbation theory in which one separates the 2dim fields

$$X^\mu(\xi) = X_0^\mu(\xi) + x^\mu(\xi)$$

background part or "expectation value" satisfying EOM

quantum part (dynamical)

Next, one expands the NLσM action around X_0^μ to get an expansion in powers of the quantum field x^μ

$\Rightarrow$ σ-model perturbation theory (ie perturbative QFT in 2dim)

To expand $G_{\mu\nu}(X)$ we use Riemann normal coordinates

Normal coordinates: $\Gamma^{\mu}{}_{\nu\rho}\big|_{x} = 0$

$$R^{\mu}{}_{\nu\sigma\rho}\big|_{x} = \partial_\sigma \Gamma^{\mu}{}_{\nu\rho}\big|_{x} - \partial_\rho \Gamma^{\mu}{}_{\nu\sigma}\big|_{x}$$

simplifies computations!

$$G_{\mu\nu}(X) = G_{\mu\nu}(X_0) - \frac{1}{3} \underbrace{R_{\mu\rho\nu\sigma}(X_0)}_{\substack{\text{Riemann tensor} \\ \text{of } M \text{ at } X_0}} x^\rho x^\sigma + \mathcal{O}(x^3)$$

We get an infinite set of coupling constants

$$\{ G_{\mu\nu}(X_0), R_{\mu\rho\nu\sigma}(X_0), \dots \}$$

Thus the σ-model expansion is given by

$$S_\sigma[X] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \Big(G_{\mu\nu}(X_0)\, \partial_a x^\mu \partial^b x^\nu \qquad \text{Kinetic term}$$

$$- \frac{1}{3} \underbrace{R_{\mu\rho\nu\sigma}(X_0)\, \partial_a x^\mu \partial^a x^\nu x^\rho x^\sigma}_{\text{quartic interaction term}} + \mathcal{O}(x^5) \Big\}$$

$\boxed{\text{interacting QFT}}$

__Remark__: what is the expansion parameter?

To be clear about the meaning of the expansion we need to expand in terms of a dimensionless parameter.

$$\left(X^\mu = X_0^\mu + x^\mu \longrightarrow X_0^\mu + \sqrt{\alpha'}\, y^\mu \right)$$

The quantum perturbation theory is in fact an expansion in powers of α'!

Note that rescaling the metric $G_{\mu\nu} \to L^2 G_{\mu\nu}$
in the Polyakov action is the same as $\alpha' \to L^{-2}\alpha'$
Then a small α' expansion corresponds to a large distance in space-time so
the dimensionless expansion parameter is $\frac{1}{r}\sqrt{\alpha'}$ with
$r \sim$ characteristic radius of curvature of target space

In spacetime we obtain an something like an EFT for a large radius expansion with cutoff $M_s \sim (\alpha')^{-1/2}$

It is a (weakly coupled) σ-model perturbation theory in the usual sense of a perturbative QFT framework, and from this one can read off the Feynman rules for diagrams.

However, generally the couplings in QFT get renormalized.
The σ-model action can be regularized by dimensional regularization but this violates scale invariance

The lack of scale invariance in a QFT is described in terms of the β-function which arises from UV divergences in Feynman diagrams.

Recall $T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta \gamma^{ab}} = 0$ in particular $T_{+-} = 0$

classically $T_{+-} = 0 \longleftrightarrow$ Weyl invariance (see next page)

At the q-level $T_{+-} \sim \frac{\delta S}{\delta \gamma} \sim \beta$-function

gets q-corrections at one-loop

We are interested in computing the (one loop) β-function to obtain conditions on the fields necessary to preserve Weyl invariance at the quantum level (i.e. $\beta = 0$)

On the tracelessness of T & Weyl invariance (from BLT)

let S be an action

$$S[\gamma, \phi]$$

metric on Σ ; collection of fields ϕ^i (for us these are the X^μ)

st Weyl action is:

$$\gamma_{ab} \to \tilde{\gamma}_{ab} = e^{2\omega}\gamma_{ab}\,; \quad \phi^i \to \hat{\phi}^i = e^{d_i\omega}\phi^i$$

$\omega = 0 + \delta\omega$

$$= \gamma_{ab} + 2\,\delta\omega\,\gamma_{ab} + \dots \qquad = \phi^i + d_i\,\delta\omega\,\phi^i + \dots$$

(for us X^μ have $d=0$)

If the action is scale inv, then

$$S[\tilde{\gamma}, \tilde{\phi}] = S[\gamma, \phi]$$

first term

so

$$0 = \delta S = \int d^2\xi \left\{ -2\frac{\delta S}{\delta\gamma^{ab}}\gamma^{ab} + \sum_i d_i \frac{\delta S}{\delta\phi^i}\phi^i \right\}\delta\omega$$

EOM for ϕ^i: $\dfrac{\delta S}{\delta\phi^i} = 0$; first term: recall $T_{ab} \propto \dfrac{\delta S}{\delta\gamma^{ab}}$ $\Rightarrow$ $\boxed{\gamma^{ab}T_{ab} = 0}$

$\gamma^{ab}T_{ab} = 0$ follows without the use of EOM for ϕ^i iff $d_i = 0$ which is the case for $S_P[X]$.

Returning to the σ-model expansion:

$$S_\sigma[X] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \Big(G_{\mu\nu}(X_0)\, \partial_a x^\mu \partial^a x^\nu$$

kinetic term

$$- \frac{1}{3} \underbrace{R_{\mu\rho\nu\sigma}(X_0)\, \partial_a x^\mu \partial^a x^\nu x^\rho}_{\text{quartic interaction term}} x^\sigma + \mathcal{O}(x^5) \Big\}$$

interacting QFT with an infinite set of couplings

We are now ready to read off Feynman rules for diagrams and do perturbation theory. Moreover we can compute one loop divergences that contribute to renormalization of couplings.

Exercise in QFT! use dimensional regularization: $2+\epsilon$ dimensions

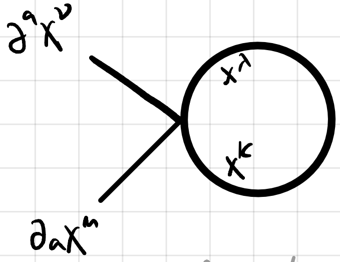
One loop logarithmic divergences in $2+\epsilon$ dims

(read off Feynman rules from S_σ)

counterterms

terms that need to be subtracted from the action to obtain a finite theory

Kinetic term



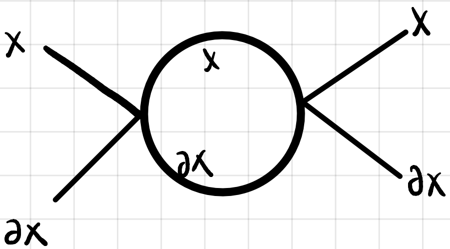
tadpole

$$R_{\mu\lambda\nu\kappa}\int \frac{d^{2+\epsilon}p}{(2\pi)^{2+\epsilon}}\frac{\eta^{\lambda\kappa}}{p^2}$$

$$-\frac{1}{2\epsilon}R_{\mu\nu}(X_0)\,\partial X^\mu \partial X^\nu$$

Ricci tensor

Interaction vertex



$$\frac{1}{9}R_{\mu\lambda\nu\sigma}R_{\rho}{}^{\lambda}{}_{\kappa}{}^{\sigma}\int \frac{d^{2+\epsilon}}{(2\pi)^{2+\epsilon}}\frac{p^2}{p^4}$$

$$-\frac{1}{18\epsilon}(R^2)_{\mu\lambda\nu\kappa}\,\partial X^\mu \partial X^\nu X^\lambda X^\kappa$$

These divergences (and those from higher loops!) can be absorbed by (not entirely easy computation) } renormalization!

⇝ a wave function renormalization of the fields x

$$X^\mu \longrightarrow X^\mu + \frac{1}{6\epsilon} R^\mu{}_\nu X^\nu + \mathcal{O}(x^2)$$

together with

- a functional renormalization of the coupling

$$G_{\mu\nu} \longrightarrow G_{\mu\nu} - \frac{1}{2\epsilon} R_{\mu\nu}$$

This gives the one-loop β-functional

← obtained from $\frac{1}{\epsilon}$ poles

$$\beta_{\mu\nu}(X) = -\frac{1}{2\pi} R_{\mu\nu}$$

The condition for conformal invariance to leading order in α' is

$$\beta_{\mu\nu} = 0 \quad \Rightarrow \quad R_{\mu\nu} = 0$$

target space must be Ricci-flat !
(space-time eq of motion)

that is, the string moves in a background spacetime which satisfies vacuum Einsteins eqs

We have recovered spacetime dynamics from a world sheet consistency condition

Higher orders in α': one gets stringy corrections to Einsteins eqs

$$\underbrace{R_{\mu\nu} + \frac{\alpha'}{2} R_{\mu\kappa\lambda\tau} R_{\nu}{}^{\kappa\lambda\tau}}_{} = 0 \quad \text{to } \mathcal{O}(\alpha'^2)$$

string theory predicts specific small corrections to Einstein's in D=26 at large radius.

Remark

Thus far we have been discussing a perturbative two dimensional QFT on the worldsheet.
Notice however that we can see an exact version developing:

$$\mathbb{R}^{1,25}_{\eta} \longrightarrow \mathbb{R}^{1,25-n}_{\eta} \times M^{n}_{G} \longrightarrow \mathbb{R}^{1,25-n}_{\eta} \times (\underset{\text{abstract}}{\mathrm{CFT}^{2d}})$$

free bosonic string
$G=\eta$

Ricci-flat
(to leading order)

non-perturbative in α'
string theory goes beyond target space
(semiclassical geom)

machinery carries over
to abstract CFT
(no ghost theorem, vertex operator formalism, etc
requires deeper study of CFTs

5.2 Including other massless modes

Apart from the graviton, we identified other massless fields in the closed string bosonic spectrum:

- The Ramond-Kalb antisymmetric field

$$B_{\mu\nu}\, dx^\mu\, dx^\nu$$

One can add to the Polyakov action the term

power counting renormalizable!

$$S^{(B)}[X] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\; \epsilon^{ab}\, B_{\mu\nu}(X)\,\partial_a X^\mu \partial_b X^\nu$$

which is reparametrization and Weyl invariant

Moreover under spacetime gauge transformations

$$B \longrightarrow B + d\Lambda \quad , \quad \Lambda \text{ a 1-form}$$

the action $S^{(B)}$ changes by a total derivative (exercise)

- The dilaton background we add

$$S^{[\Phi]}[X;\gamma] = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \Phi(X)\, R^{(2)}(\gamma)\alpha'$$

[We had previously ignored this term for Φ = constant as in this case the integrand is a total derivative.]

The integrand not a total derivative if $\Phi \neq$ constant

This term however is not Weyl invariant:

$$\gamma \to e^{2\omega(\sigma)}\gamma \Rightarrow R^{(2)} \to e^{-2\omega}(R^{(2)} - 2\nabla^2\omega)$$

One can show however that a classical Weyl variation of S^{Φ} can be cancelled by an $\mathcal{O}(\alpha')$ variation of $S^{(G)} + S^{(B)}$!

$$S = S^{(G)} + S^{(B)} + S^{(\Phi)}$$

An involved computation of the β-functional extending the one-loop computation for $S^{[G]}$ gives for the full σ-model action $S^{[G]} + S^{[B]} + S^{[\Phi]}$:

$$\beta^G_{\mu\nu} = \alpha' \Big(\underbrace{R_{\mu\nu} - \tfrac{1}{4} H_{\mu\lambda\sigma} H_\nu{}^{\lambda\sigma}}_{\text{1-loop } G+B} + 2 \underbrace{D_\mu D_\nu \Phi}_{\text{classical } \Phi} \Big)$$

$H = dB$

$H_{\mu\nu\rho} = 3\,\partial_{[\mu} B_{\nu\rho]}$

$$\beta^B_{\mu\nu} = \alpha' \Big(\underbrace{-\tfrac{1}{2} D^\lambda H_{\lambda\mu\nu} + D^\lambda \Phi\, H_{\lambda\mu\nu}}_{\text{1-loop } G+B} \Big)$$

D_μ = covariant derivative on space time

$$\beta^\Phi = \underbrace{\tfrac{1}{6}(D-26)}_{\text{1-loop } G+B} + \alpha' \Big(\underbrace{(D_\mu \Phi)(D^\mu \Phi)}_{\text{1-loop } \Phi} - \underbrace{\tfrac{1}{2} D^2 \Phi - \tfrac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}}_{\text{two loop } G+B} \Big)$$

references: Friedan's thesis; Callan & Thorlacius "Sigma models & string theory"; Tseytlin "Conformal anomaly in a 2dim σ-model"

Next: strings in background fields continued.

Space-time effective action