STRING THEORY I

Lecture 13

$\boxed{5}$ Strings in background fields

5.1 Background field expansion and the Weyl anomaly

5.2 Including other massless modes

5.3 Spacetime effective actions

5.4 The dilaton revisited

5.5 Energy scales

Last lecture:

constructed a 2 dim NLσM

$$S = S^{(G)} + S^{(B)} + S^{(\Phi)}$$

general 2 dim QFT which is reparametrization invariant (& renormalizable)

$S^{(G)} + S^{(B)}$ is classically Weyl invariant

but $S^{(\Phi)}$ is not (unless Φ = constant)

S is hard to analyze because couplings ($G_{\mu\nu}$, $B_{\mu\nu}$ & Φ) depend on X

$\Rightarrow$ We analyzed the action S in terms of the background field expansion

$$\underset{\text{soln of classical EOM}}{X^\mu = X_0^\mu} + x^\mu$$

x^μ ~ quantum fluctuation

This gives a perturbative expansion of the NLσM in powers of the fluctuations. α'

Crucially we demanded that the resulting 2dim QFT be Weyl invariant at the quantum level.

This requirement leads to the computation of the β-functional. The preservation of the Weyl symmetry at the quantum level, ie

$$\beta^{(G)} = 0 \quad , \quad \beta^{(B)} = 0 \quad , \quad \beta^{(\Phi)} = 0$$

then imposes constraints on the spacetime fields G, B & Φ which are interpreted as EOM for these fields (eg to first order in α' $R_{\mu\nu} = 0$, etc)

Recall [lecture 12]

An involved computation of the β-functional extending the one-loop computation for $S^{[G]}$ gives for the full σ-model action $S^{[G]} + S^{[B]} + S^{[\Phi]}$:

$$\beta^{G}_{\mu\nu} = \alpha' \Big(\underbrace{R_{\mu\nu} - \frac{1}{4} H_{\mu\lambda\sigma} H_{\nu}{}^{\lambda\sigma}}_{\text{1-loop } G+B} + \underbrace{2 D_\mu D_\nu \Phi}_{\text{classical } \Phi} \Big)$$

$H = dB$

$H_{\mu\nu\rho} = 3\, \partial_{[\mu} B_{\nu\rho]}$

$$\beta^{B}_{\mu\nu} = \alpha' \Big(\underbrace{-\frac{1}{2} D^\lambda H_{\lambda\mu\nu} + D^\lambda \Phi H_{\lambda\mu\nu}}_{\text{1-loop } G+B} \Big)$$

D_μ = covariant derivative on space time

$$\beta^{\Phi} = \underbrace{\frac{1}{6}(D-26)}_{\text{1-loop } G+B} + \alpha' \Big(\underbrace{(D_\mu \Phi)(D^\mu \Phi)}_{\text{1-loop } \Phi} - \underbrace{\frac{1}{2} D^2 \Phi - \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}}_{\text{two loop } G+B} \Big)$$

references: Friedan's thesis; Callan & Thorlacius "Sigma models & string theory"; Tseytlin "Conformal anomaly in a 2dim σ-model"

5.3 Space-time effective action

We want to interpret the vanishing of the β-function as spacetime equations of motion.

Indeed, one can show that they arise as the Euler-Lagrange equations for the effective action

$$S_{26}^{S} = \frac{1}{2K_0^2}\int d^{26}x\,\sqrt{-G}\,e^{-2\Phi}\left(R - \frac{1}{12}|H|^2 + 4|D\Phi|^2\right)$$

"string frame" action (G, B, Φ in S_{26}^{S} are the fields that appear in the σ-model action)

K_0 related to Newton's constant: see next

For space-time computations one often uses the "Einstein frame":

let $\quad \tilde{\Phi} = \Phi - \Phi_0 \quad , \quad \tilde{G} = e^{\frac{1}{6}\tilde{\Phi}} G$

$$S_{16}^{(E)} = \frac{1}{2\kappa^2} \int d^{16}x \sqrt{-\tilde{G}} \left(\tilde{R} - \frac{1}{12} e^{-\frac{1}{3}\tilde{\Phi}} |H|^2 - \frac{1}{6} |D\tilde{\Phi}|^2 \right)$$

$\kappa = \kappa_0 e^{\Phi_0}$

indices raised and lowered with $\tilde{G}$

Einstein-Hilbert term takes the canonical form with gravitational coupling
$\kappa = (4\pi G_N)^{1/2}$

$M_s = 1/\sqrt{\alpha'}$

The spacetime action should capture the classical limit when $E \ll M_s$. The stringy corrections to this can be seen from the corrected β function

$$\beta = \beta^{(0)} + \alpha' \beta^{(1)} + (\alpha')^2 \beta^{(2)} + \cdots$$

harder ..

(For example $\beta^{G}_{\mu\nu} \sim \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\mu\nu\lambda\sigma} R_{\nu}{}^{\lambda\rho\sigma} + \cdots$)

The corrected β-functional gets interpreted as Euler-Lagrange equations for an α'-corrected action

$$S_{26} = S_{26}^{(0)} + \alpha' S_{26}^{(1)} + (\alpha')^2 S_{26}^{(2)} + \cdots$$

EFT (expansion with cutoff scale M_s)

$\frac{1}{M_s^2}$

4-derivative terms

$\frac{1}{M_s^4}$

6-derivative terms

↳ effective action obtained after integrating out massive modes

15.4 The dilaton revisited

Recall

$$S^{\Phi} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\gamma}\, R^{(2)}(\gamma)\, \Phi(X)$$

Previously we had ignored this term for Φ = constant because then the integrand is a total derivative. This however is not the right way to look at this term in an interacting theory in a background with $\Phi = \Phi_0$ = constant.

Theorem (differential topology): Gauss-Bonnet theorem

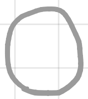
Let Σ be a 2-dim surface. Then

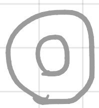
If $\Sigma = \Sigma_g$ is a compact Riemann surface with no boundaries and genus g (# of handles)

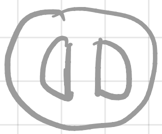
eg  Σ_2

$$\frac{1}{4\pi}\int_{\Sigma_g} dA \, R^{(2)} = \chi(\Sigma_g) = 2-2g$$

If $\Sigma = \Sigma_{g,h}$ is a Riemann surface with genus g and h boundaries

eg  disk $g=0, h=1$, $\chi = 1$

annulus  $\chi=0$, $g=0, h=2$

 $\chi=1$, $g=0$ $h=3$

$$\frac{1}{4\pi}\int_{\Sigma_{g,h}} dA \, R^{(2)} + \frac{1}{2\pi}\int_{\partial\Sigma_{g,h}} ds\, K = \chi(\Sigma_{g,h}) = 2-2g-h$$

K: extrinsic curvature

ie independent of γ

χ = Euler characteristic (topological invariant)
for a surface Σ with g handles and h boundaries

This means that in a background $\boxed{\Phi = \Phi_0}$

$$S^{\Phi} = \Phi_0 \chi_{g,h}$$

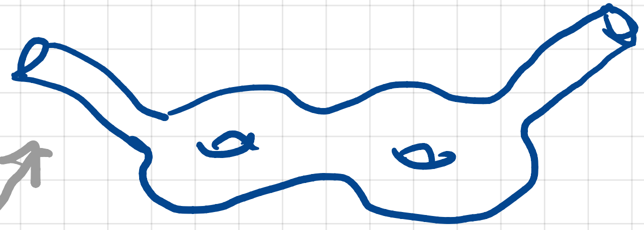
where $\chi_{g,h}$ is determined by this theorem and depends on g and h.

Clearly this term in the total action

$$S = S^{(a)} + S^{(B)} + S^{\Phi}$$

adds to the amplitudes a factor of $e^{-\Phi_0}$ to some power determined by χ.

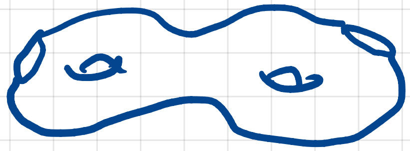
$\boxed{\text{Closed}}$ string amplitudes:



$\longleftrightarrow$

topologically the same

$g=2, \quad h=2$



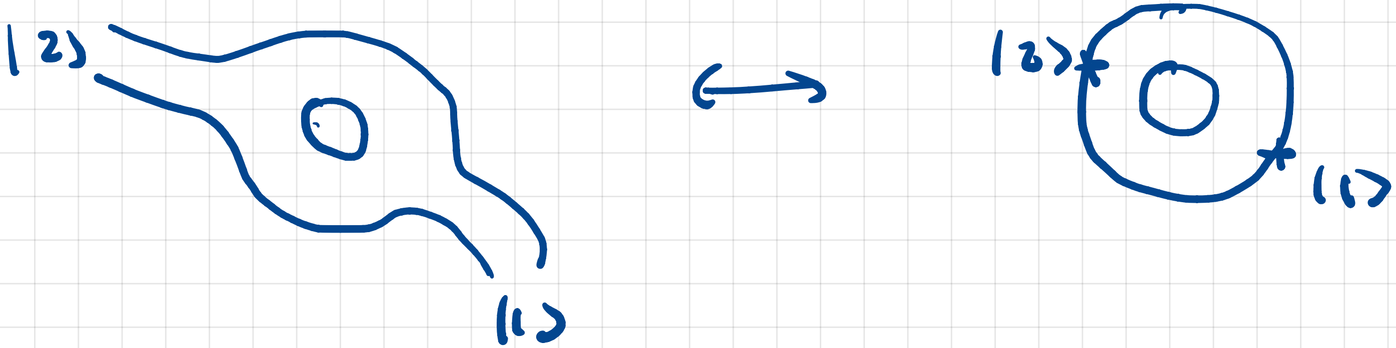
$\chi = 2 - 2g - h$

incomming & outgoing states introduce extra boundaries

each loop gets a factor $e^{-2\Phi_0}$

each external closed string gets a factor $e^{-\Phi_0}$

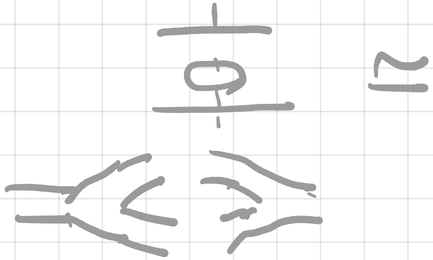
Open string amplitudes



each open string **loop** gets a factor $e^{-\Phi_0}$

↳ adds a boundary 

each **external** open string gets a factor $e^{-\Phi_0/2}$



$\simeq$ 2 open string VOp

These factors match powers of the open string (g_o) & closed string (g_c) coupling constants. Consider shifting the dilaton Φ by $\Phi \to \Phi + c$, where c is an arbitrary constant, corresponds to a rescaling of these couplings

$e^{-S\Phi} \to e^{-S\Phi} e^{-\chi c}$

$$g_c \longrightarrow g_c e^{-c} \qquad g_o \longrightarrow g_o e^{-c/2}$$

so the value of g_c (or g_o) can be absorbed into a shift in the expectation value of Φ.

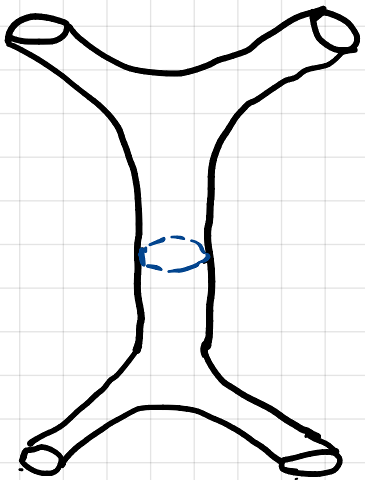
This means that the string coupling constants are **not** parameters of the theory, and are given in terms of the expectation value Φ_0 of Φ (in fact they are dynamical)

$\Phi(X) = \Phi(x_0 + x) = \Phi_0 + \cdots$

Recall that from the computation of the annulus amplitude we had $g_o^2 \sim g_c$

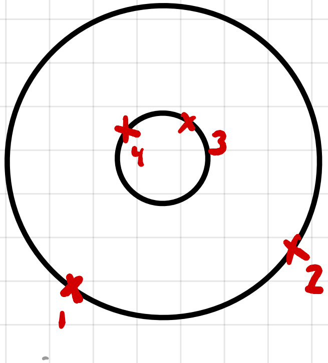
(lecture #12 but in lecture #11 notes)

$\sim g_c^2$



tree level closed string amplitude

$\sim$



$\sim g_o^4$

one loop open string amplitude

a single geometry with two interpretations

Consistency : $g_c \sim g_o^2$

One can compute the precise normalization factor

$$g_o^2 = 2^{18} \pi^{25/2} (\alpha')^6 g_c$$

(Polchinski exercise 7.8)

These considerations illustrate the fact that in string theory there are no continuous parameters.

Parameters appear then as expectation values of dynamical spacetime fields.

5.5 Energy scales

↳ Observations about the energy scales involved in the space-time effective action obtained by requiring that its EOM are the same as the vanishing of the Beta functions.

For the one-loop Beta function we obtained the effective action to leading order in α'. We found in the string frame

$$S_{26}^{(S)} = \frac{1}{2\kappa_0^2} \int d^{26}x \sqrt{-G}\, e^{-2\Phi} \left(R - \frac{1}{12}|H|^2 + 4|D\Phi|^2 \right)$$

$H = dB$

while in the Einstein frame

$$S_{26}^{(E)} = \frac{1}{2\kappa^2} \int d^6x \sqrt{-\tilde{G}} \left(\tilde{R} - \frac{1}{12} e^{-\frac{1}{3}\tilde{\Phi}} |H|^2 - \frac{1}{6}|D\tilde{\Phi}|^2 \right)$$

where $\tilde{G}_{\mu\nu} = e^{\frac{1}{6}\tilde{\Phi}} G_{\mu\nu}$, $\tilde{\Phi} = \Phi - \Phi_0$, $\kappa = \kappa_0 e^{\Phi_0}$

(and indices are raised & lowered with $\tilde{G}$)

The Einstein frame is constructed such that the Einstein-Hilbert term takes the canonical form with gravitational coupling

$$\kappa = (8\pi G_N)^{1/2}$$

- The gravitational coupling

$$\kappa = \kappa_0 e^{\Phi_0} = (8\pi G_N)^{1/2} \sim (M_{Pl})^{-\frac{1}{2}(D-2)}$$

controls spacetime quantum effects

- We also have the string scale

$$\alpha' \sim M_s^{2}$$

(Recall that we obtained an effective theory from the NLσM large radius expansion with cutoff M_s)

The string scale controls stringy corrections (world sheet quantum corrections).

▶ The gravitational coupling κ and string scale α' are related by the expectation value of the dilaton (e^{Φ_0})

We have a dimensionless ratio

$$\frac{M_s}{M_{pl}} \sim e^{\frac{2}{D-2}\Phi_0}$$

$e^{\Phi_0} \to 0$ gives a classical limit in space time

So we have the effective action for energies $E \ll M_s$ in the limit $\frac{M_s}{M_{pl}} \to 0$ (suppress space time quantum effects)

Next: compactifications

↳ illustrate · $\mathbb{R}^{1,24} \times S^1_R$

· T-duality