A.6 Renewal theory, insurance ruin and queueing theory (Optional)

This sheet is intend to be some extra vacation work, we will not have time to cover it in classes.

1. Potential customers arrive at a single-server bank according to a Poisson process $(N_t)_{t\geq 0}$ with rate λ. However, potential customers will enter the bank only if the server is free when they arrive, and otherwise will go home. Assume that the service times are independent random variables with probability density function g and mean ν.

   (a) Denote by X_t the number of customers that *have left after completed service* before time t, $t \geq 0$. Show that $(X_t)_{t\geq 0}$ is a renewal process, and describe its inter-renewal distribution.

   (b) Calculate the asymptotic rate $\lim_{t\to\infty} X_t/t$ at which customers leave the bank (after completed service).

   (c) Consider the proportion $P_t = X_t/N_t$. What long-term proportion of potential customers are actually served?

   (d) Consider the sequence of departure times T_n, $n \geq 1$ (departures after completed service). What long-term proportion of time is the server busy?

   Hint: Consider this proportion at departure times first and then argue as in the proof of the strong law of renewal theory.

2. In the setting of the previous sheet Question 4 (Proof of the Ergodic Theorem), suppose that the jump chain is also positive recurrent, denote by η its stationary distribution and by $\Pi = (\pi_{i,j})_{i,j\in\mathbb{S}}$ its transition matrix. Let $X_0 = i$.

   Denote by $N(t)$ the number of transitions of X up to time $t \geq 0$, and by $N_i(t)$ the number of transitions to i up to time $t \geq 0$. Show that $\dfrac{N_i(t)}{N(t)} \to \eta_i$ almost surely as $t \to \infty$. Also show that $\dfrac{N_i(t)}{t} \to \dfrac{1}{m_i}$ almost surely as $t \to \infty$. Deduce that $\dfrac{N(t)}{t} \to \dfrac{1}{m_i\eta_i}$ almost surely, and that the limit does not depend on $i \in \mathbb{S}$.

3. Consider a single-server queueing system with Poisson arrivals at rate λ and exponential service times at rate μ. The system has the following special feature: the server can serve two customers at the same time. He can also serve a single customer in the system but then a second customer cannot be jointly served before the single customer leaves. Take $\mathbb{S} = \mathbb{N} \cup \{\emptyset\}$. Let $X_t = \emptyset$ if the server is idle. Let $X_t = 0$ if the server is busy but no-one else is waiting to be served. If the server is busy and there are n people waiting to be served, set $X_t = n$.

   (a) Determine the Q-matrix and the invariant distribution.

   Hint: Try $\xi_n = \alpha^n \xi_0$ for $n \in \mathbb{N}$.

   (b) Determine the long-term proportion of customers that are served alone.

   Hint: Which transitions correspond to a beginning single service? Consider the counting processes counting these transitions separately.

4. A branch of an insurance company has at its disposal an initial capital of $u > 0$ at time $t = 0$ and receives linear premium income, ct by time $t \geq 0$, from which it has to meet claims A_n, $n \geq 1$, of independent exponential sizes with parameter μ, arriving at the times T_n, $n \geq 1$, of a Poisson process $(N_t)_{t\geq 0}$ with rate λ. Denote the reserve at time t by R_t.

(a) Using $Z_i = R_{\varepsilon i} - R_{\varepsilon(i-1)}$, $i \geq 1$, or otherwise, show that

$$\frac{R_{\varepsilon n}}{\varepsilon n} \to c - \frac{\lambda}{\mu} \qquad \text{almost surely as } n \to \infty,$$

for any $\varepsilon > 0$ fixed, and hence that $R_{\varepsilon n} \to \infty$ almost surely, if $c > \lambda/\mu$.

(b) Denote by Y_n, $n \geq 1$, the inter-renewal times of the claims, and define $T_0 = 0$, $T_n = Y_1 + \cdots + Y_n$. Define $S_n = R_{T_n}$ and also consider $R_{T_n-} = S_n + A_n$. Show that

$$\frac{R_{T_n}}{n} \to \frac{c}{\lambda} - \frac{1}{\mu} \qquad \text{and} \qquad \frac{R_{T_n-}}{n} \to \frac{c}{\lambda} - \frac{1}{\mu} \qquad \text{almost surely as } n \to \infty$$

(c) Using $R_{T_{N_t}} \leq R_t \leq R_{T_{N_t+1}-}$, deduce from (b) that

$$\frac{R_t}{t} \to c - \frac{\lambda}{\mu} \qquad \text{almost surely as } t \to \infty.$$

5. Consider a stationary $M/M/1$ queue $(X_t)_{t\geq 0}$ with independent exponential inter-arrival times with rate λ and independent exponential service times with rate μ. Here, the initial distribution is ξ with $\xi_i = \rho^i(1-\rho)$, where $\rho = \lambda/\mu < 1$. Denote by $T_0 = 0$ and $T_{n+1} = \inf\{t > T_n : X_t \neq X_{T_n}\}$ the jump times, by $M_n = X_{T_n}$ the embedded jump chain. Denote by $A_0 = 0$, $A_{m+1} = \inf\{t > A_m : X_t - X_{t-} = 1\}$, $m \geq 0$, the successive arrival times.

(a) Show that $\mathbb{P}(A_1 > T_k | X_0 = m) = \left(\frac{\mu}{\lambda + \mu}\right)^k$, $m \geq k$.

(b) Show that $\mathbb{P}(A_1 > T_k) = \left(\frac{\lambda}{\lambda + \mu}\right)^k$, $k \geq 0$.

(c) Show that $\mathbb{P}(M_k = i | A_1 > T_k) = \rho^i(1-\rho)$, $i \geq 0$.

(d) Show that $\mathbb{P}(X_{A_1} = i) = \rho^i(1-\rho)$, $i \geq 2$. Without any further calculations, is ξ stationary for $(X_{A_m})_{m\geq 0}$?

6. Let $\widetilde{X}$ be a delayed renewal process whose first renewal time has density g, the subsequent inter-renewal times density f. Let $F(t) = \int_0^t f(s)ds$, $G(t) = \int_0^t g(s)ds$ and $\overline{F}(t) = 1 - F(t)$, $\overline{G}(t) = 1 - G(t)$.

(a) Show that $\widetilde{m}(t) = \mathbb{E}(\widetilde{X}_t)$ (as opposed to the undelayed $m(t) = \mathbb{E}(X_t)$) satisfies both

$$\widetilde{m} = G + m * g \quad \text{and} \quad \widetilde{m} = G + \widetilde{m} * f.$$

(b) Show, now by conditioning on the last arrival before time t that

$$\mathbb{P}(\widetilde{E}_t > y) = \overline{G}(t+y) + \int_0^t \overline{F}(t+y-x)\widetilde{m}'(x)dx.$$

(c) If more specifically $g(y) = f_0(y) = \overline{F}(y)/\mu$, show that $\widetilde{m}(t) = t/\mu$ and that $\widetilde{E}_t$ also has density f_0, for all $t \geq 0$.