

1. Let G be a graph in which every vertex has even degree. Show using the handshaking lemma that G cannot contain any bridges.
2. Find graphs G_1 and G_2 overlapping in two vertices which are not adjacent in either graph such that $\chi(G_1 \cup G_2) > \max(\chi(G_1), \chi(G_2))$. How big can the difference be?
3. Let G be a connected graph with n vertices and let v be a given vertex of G . Show that we can list the vertices as $v_1, v_2, \dots, v_{n-1}, v_n = v$ so that each vertex other than v has a neighbour coming after it in the list.
4. Show that $e(G) \geq \binom{\chi(G)}{2}$ for every graph G .
5. Let G be a graph of order n , and write \overline{G} for the complement of G . Prove that $\chi(G)\chi(\overline{G}) \geq n$ and deduce that $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$.
6. Suppose that $\chi(G) = k$ and $c : V(G) \rightarrow \{1, \dots, k\}$ is a proper k -colouring of G . Must there be a path $x_1 \cdots x_k$ in G with $c(x_i) = i$ for each i ?
7. (a) Show that for every graph G there is an ordering of $V(G)$ for which the greedy algorithm uses $\chi(G)$ colours.
(b) Find a bipartite graph and an ordering of its vertices so that the greedy algorithm uses 2017 colours.
8. (a) Prove that if T is a tree with n vertices then its chromatic polynomial is $p_T(x) = x(x-1)^{n-1}$.
(b) Show that the chromatic polynomial of C_n is given by $p_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$ for each $n \geq 3$.
(c) For $n \geq 3$, the *wheel* W_{n+1} is the graph on $n+1$ vertices obtained from C_n by adding a new vertex adjacent to everything. Calculate the chromatic polynomial of W_{n+1} .
9. Let G be a graph of order $n \geq 3$ and let $p_G(x) = \sum_{i=0}^{n-1} (-1)^i a_i x^{n-i}$ be the chromatic polynomial of G . Recall that $a_0 = 1$, $a_1 = e(G)$. Show that $a_2 = \binom{e(G)}{2} - t(G)$, where $t(G)$ is the number of copies of K_3 in G .
10. Find graphs G and H such that $|G| = |H|$, $e(G) = e(H)$, $\chi(G) > \chi(H)$ and $p_G(x) > p_H(x)$ for all sufficiently large x .

Optional bonus questions. These may not be covered in classes; MFOCS students should attempt them!

11. For which k and ℓ can you construct a graph G with $\chi(G) = k$ and $\omega(G) = \ell$?
12. An *acyclic orientation* of G is a way of assigning a direction $u \rightarrow v$ or $v \rightarrow u$ to each edge uv of G so that there is no cycle $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_t \rightarrow v_1$. Show that $|p_G(-1)| = (-1)^{|G|} p_G(-1)$ is the number of acyclic orientations of G .
13. Let G be the infinite graph with vertex set \mathbb{R}^2 in which two vertices are joined if and only if they are at Euclidean distance 1. Prove that $4 \leq \chi(G) \leq 7$.

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk