STRING THEORY I

Lecture 14

$\boxed{6}$ # Compactifications

Consider S^1-compactifications of the bosonic string

$$\mathbb{R}^{1,25} \longrightarrow \mathbb{R}^{1,24} \times S^1_R$$

circle of radius R

fields $\quad X^\mu \begin{cases} X^i & i = 0, \dots, 24 \\ X^{25} \sim X^{25} + 2\pi R \end{cases}$

parametrizes circle of radius R

We will discuss this from our two perspectives

① From the spacetime EFT

↳ Kaluza-Klein mechanism to obtain an effective theory on $\mathbb{R}^{1,24}$

② From the world sheet CFT perspective

target space will "look" the same as for flat $\mathbb{R}^{1,25}$ but with non-trivial topology

BLT

(see D Tong lecture notes)

$\boxed{6.1}$ Spacetime EFT approach for the closed bosonic string theory

⇝ Kaluza-Klein ansatz for the fields to obtain an effective action in (1,24)-dimensions.

Fields $X^\mu \begin{cases} X^i & i = 0, \dots, 24 \\ X^{25} & \text{coordinate on } S^1_R \end{cases}$

metric
$$G_{\mu\nu} dX^\mu dX^\nu = \underbrace{G_{ij}}_{G_{ij}(X^i, X^{25})} dX^i dX^j + e^{2\sigma} (dX^{25} + A_i dX^i)^2$$

$$(\text{so } G_{25,25} = e^{2\sigma} \ , \ G_{25,i} = e^{2\sigma} A_i)$$

2K field
$$B_{\mu\nu} dX^\mu \wedge dX^\nu = B_{ij} dX^i \wedge dX^j + \tilde{A}_i dX^i \wedge dX^{25}$$

$$(\text{so } B_{i25} = \tilde{A}_i)$$

dilaton
$$\Phi = \Phi_{(25)} + \frac{1}{2} \sigma$$

- One then rewrites the effective action $S_{(26)}$ in terms of

$$G_{ij},\ A_i,\ e^{2\sigma}$$
$$B_{ij},\ \tilde{A}_i$$
$$\Phi_{(25)}$$

- This is a long computation, but that is ok.

Recall
$$S^S_{26} = \frac{1}{2\kappa_0^2}\int d^{26}x\,\sqrt{-G}\,e^{-2\Phi}\left(R - \frac{1}{12}|H|^2 + 4|\nabla\Phi|^2\right)$$

For example:

$$R(G_{26}) = R(G_{25}) - \frac{1}{2}e^{2\sigma}\,F(A)^{ij}F(A)_{ij} - 2e^{-\sigma}\nabla^i\nabla_i e^{\sigma}$$

etc

All these fields depend on X^i but also on X^{25}

Due to the identification $X^{25}(\tau,\sigma) \sim X^{25}(\tau,\sigma) + 2\pi R$

we can expand these fields in Fourier modes with with respect to X^{25}:

$$F(X^i, X^{25}) = \sum_{n\in\mathbb{Z}} F_n(X^i)\, e^{in\frac{1}{R}X^{25}}$$

- Finally we integrate $S_{[26]}$ over X^{25} to obtain a theory in 25-dimensions

We will not be able to do all this explicitly (but see below for the dilaton)

↳ long computation indeed !

as we will see

$n=0$ in Fourier series for the fields

Note however (see below) that the zero modes typically give the massless sector of the theory

These zero modes are:

metric $G_{ij}(x^i)$

KR field $B_{ij}(x^i)$

2x 1-form gauge fields A and $\tilde{A}$

what is the gauge symmetry?

A (graviphoton) & $\tilde{A}$ (KR-photon)
correspond to $U(1) \times U(1)$ gauge fields
(A: symmetry descends from 26-dim diffeomorphism
$\tilde{A}$: exercise)

2 scalars: σ, $\Phi_{(25)}$

Let's look at the dilaton more carefully.

$$\Phi(X^\mu) = \Phi(X^i, X^{25})$$ recall $X^{25} \sim X^{25} + 2\pi R$

We can expand this field (and any other fields) in Fourier modes with respect to X^{25}:

$$\Phi(X^\mu) = \sum_{n\in\mathbb{Z}} e^{in\frac{1}{R}X^{25}} \underbrace{\phi_n(X^i)}_{\text{independent of } X^{25}}$$

$$\underbrace{\phi_n = \phi_{-n}^*}_{\text{because } \Phi \text{ is real-valued}}$$

Dilaton terms in the action $S_{(26)}$: (ignoring coupling to gravity)

$$|\nabla_{26}\Phi|^2 = \partial_i\Phi\,\partial^i\Phi + (\partial_{25}\Phi)^2$$

$$= \cdots = \sum_{n,m} e^{i(n+m)\frac{1}{R}X^{25}} \underbrace{\left\{\partial_i\phi_n\,\partial^i\phi_m - \frac{nm}{R^2}\phi_n\phi_m\right\}}_{\text{Independent of } X^{25}}$$

Then

in principle there is a factor $\sqrt{G_{26}}$ (ignore)

$$\int d^{26}X \, |\nabla_{26} \Phi|^2 = \int d^{25}X \, 2\pi R \sum_{n=-\infty}^{\infty} \Big\{ \partial_i \phi_n \partial^i \phi_{-n} + \frac{n^2}{R^2} \underbrace{\phi_n \phi_{-n}}_{|\phi_n|^2} \Big\}$$

$\Rightarrow$ the massless dilaton $\Phi(X^M)$ of the 26-dimensional EFT gives rise to a discrete infinite tower of scalar fields ϕ_n, the Kaluza-Klein modes, with mass $M_n^2 = \frac{1}{R^2} n^2$

For small R all are heavy modes except the massless mode ($n=0$)

can ignore for distance scales $\gg R$

Now, note that under a spacetime diffeomorphism

$$\delta X^\mu = \epsilon^\mu(X)$$

the metric changes as

$$\delta G_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu .$$

Thus under

$$\delta X^{25} = \epsilon(X^i)$$

reparametrisation of X^{25} direction

we find

$$\delta A_i = \partial_i \epsilon$$

$\left(A_i = G_{25,i} \Rightarrow \delta G_{25,i} = \partial_i \epsilon \right)$

So indeed we interpret A_i as a $U(1)$ gauge field and the gauge symmetry descends from the 26-dimensional diffeomorphism invariance.

A_i is called the graviphoton.

The massive KK modes Φ_n $(n \neq 0)$ are charged under this gauge field A_i

$$\Phi(X^\mu) \longrightarrow \sum_{n \in \mathbb{Z}} e^{in\frac{1}{R}(X^{25}+\epsilon)} \Phi_n(X^i)$$

$\frac{n}{R} = p^{25}$

hence $\phi_n \longrightarrow e^{in\epsilon/R} \phi_n$

That is the KK-momentum is charge for the graviphoton.

One can show that there are no excitations charged under the $U(1)$ symmetry associated to the Ramond-Kalb-photon.

Massless sector of the effective 25-dimensional theory:

$G_{\mu\nu}(X) \longrightarrow G_{\mu\nu}(X^i) : \{ G_{ij}(X^i),\ G_{i,25}(X^i),\ G_{25,25}(X^i) \}$

25 dim graviton

graviphoton $e^{2\sigma} A$

radion $e^{2\sigma}$

$B_{\mu\nu}(X) \longrightarrow B_{\mu\nu}(X^i) : \{ B_{ij}(X^i),\ B_{i,25}(X^i) \}$

25 dim KR field

KR - photon $\tilde{A}$

$\Phi(X) \longrightarrow \Phi(X^i)$ 25 dim dilaton

✓

Remark: we have introduced a new scale

$$M_{KK} \sim \frac{1}{R}$$

We should not trust the EFT analysis for $M_{KK} \sim M_s$ (small radius R).

However, in this case one can perform an exact analysis of the worldsheet CFT.

6.2 World-sheet perspective (Closed string)

The target space for the two dimensional NLσM is $\mathbb{R}^{1,25}$ however X^{25} is a field on S^1_R which is periodic ie $X^{25} \sim X^{25} + 2\pi R$

This non trivial topology has very interesting consequences.

① a space-time translation by $2\pi R$: $e^{2\pi i R \hat{P}_{25}}$ ← generates translations by x^{25} should act as identity:

$$e^{2\pi i R \hat{P}_{25}} |\cdots, k_{25}\rangle = e^{2\pi i R k_{25}} |\cdots, k_{25}\rangle = |\cdots, k_{25}\rangle$$

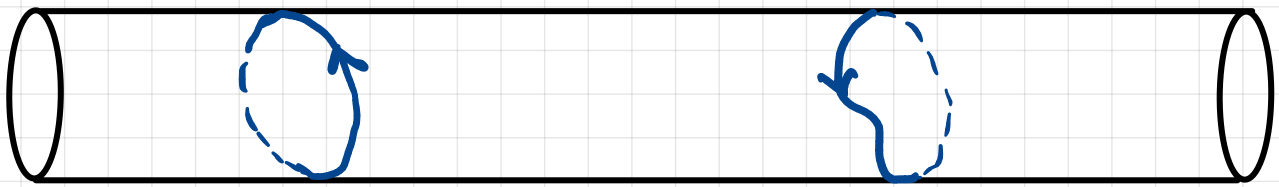
iff $\boxed{k_{25} = \frac{m}{R}}$ $m \in \mathbb{Z}$ just as in the EFT analysis

② $X^{25}(\tau, \sigma+\pi) = X^{25}(\tau,\sigma) + 2\pi R w \qquad w \in \mathbb{Z}$

(that is X^{25} only needs to be periodic $\sigma \to \sigma+\pi$ up to $2\pi R$ shifts)

w is called the winding number

Term $2\pi R w$ gives rise to closed strings wrapped on S^1_R and counts how many times the string wraps around S^1_R



$w = 1$ $\qquad\qquad$ $w = -1$

winding is a stringy effect: there is nothing like this in the EFT we discussed

(In the 26 dim EFT: these are solitons!)

Spectrum of the string with target space $\mathbb{R}^{1,24} \times S^1_R$

Mode expansion of X^{25} (which respects $X^{25}(\tau, \sigma+\pi) = X^{25}(\tau,\sigma) + 2\pi R w$)

$$X^{25}(\tau,\sigma) = x^{25} + \tau p^{25} + 2 w R \sigma + \frac{i}{2} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^{25} e^{-2in\sigma_-} + \tilde{\alpha}_n^{25} e^{-2in\sigma_+}\right)$$

$$= X_R^{25}(\sigma_-) + X_L^{25}(\sigma_+)$$

where

$$X_R^{25}(\sigma_-) = x_R^{25} + \frac{1}{2} p_R^{25} \sigma_- + \frac{i}{2} \sum_{n\neq 0} \frac{1}{n} \alpha_n^{25} e^{-2in\sigma_-}$$

$$X_L^{25}(\sigma_+) = x_L^{25} + \frac{1}{2} p_L^{25} \sigma_+ + \frac{i}{2} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^{25} e^{-2in\sigma_+}$$

and $\quad p_L^{25} = p^{25} + 2Rw \qquad p_R^{25} = p^{25} - 2Rw$

$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$

This is just as in $\mathbb{R}^{1,25}$ except that $\alpha_0^{25} = \frac{1}{2} p_R^{25}$, $\tilde{\alpha}_0^{25} = \frac{1}{2} p_L^{25}$

$(\alpha_0^{25} + \tilde{\alpha}_0^{25} = p^{25} \; ; \; \alpha_0^{25} - \tilde{\alpha}_0^{25} = -2Rw)$

The mode expansion of X^i $\quad i = 0, \ldots, 24$ remains unchanged.

The string states are similar to those of $\mathbb{R}^{1,25}$ except that now we have quantised KK-modes and winding on the circle:

(Recall in $\mathbb{R}^{1,25}$ we had $\prod \alpha^{\mu}_{-n} \prod \tilde{\alpha}^{\nu}_{-m} |0; k\rangle$)

$$\prod \alpha^{i}_{\underline{n}} \prod \tilde{\alpha}^{j}_{\underline{m}} |k; m; w\rangle$$

25 dim momentum

$p_{25} = \frac{m}{R}$

from $X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R w$

Physical spectrum: Virasoro operators & constraints

$$L_0 = \frac{1}{2}\left[\alpha_0 \cdot \alpha_0 + (\alpha_0^{25})^2\right] + \underbrace{\sum_{n>0} \alpha_{-n} \cdot \alpha_n + \sum_{n>0} \alpha_{-n}^{25}\, \alpha_n^{25}}_{N}$$

(1+24) dim inner product

$$L_m = \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n + \frac{1}{2} \sum \alpha_{m-n}^{25}\, \alpha_n^{25}$$

Similar expressions for $\tilde{L}_m$

Mass-shell and level matching conditions

$$(L_0 - 1)|\phi\rangle = 0 \qquad (\tilde{L}_0 - 1)|\phi\rangle = 0$$

$$L_0 - 1 = \frac{1}{8}\left(p_{25}^2 + p_R^2\right) + N - 1 \longrightarrow M_{25}^2 = p_R^2 + 8(N-1)$$

$p_R = p_{25} - 2Rw$

$$\tilde{L}_0 - 1 = \frac{1}{8}\left(p_{25}^2 + p_L^2\right) + \tilde{N} - 1 \longrightarrow M_{25}^2 = p_L^2 + 8(\tilde{N}-1)$$

$p_L = p_{25} + 2Rw$

Then:

$$M_{(25)}^2 = \frac{m^2}{R^2} + \underbrace{4R^2w^2} + 4(N + \tilde{N} - 2)$$

mass shell condition

contribution to the mass from momentum along the compact direction

contribution to the mass of the string winding around the circle $|w|$ times $\left(\left|\frac{Rw}{\alpha'}\right|^2 = (2\pi R T w)^2\right)$

$$N - \tilde{N} = mw$$

level-(mis)matching condition

For $w = 0$ this matches results from EFT

lowest energy state: tachyon: $N = \tilde{N} = 0$, $m = 0$, $w = 0$: $M_T^2 = -8$

Massless spectrum: for $N = \tilde{N} = 1$ ($\Rightarrow mw = 0$)

$m = w = 0$

25-dim graviton: $\gamma_{ij}\, \alpha^i_{-1} \tilde{\alpha}^j_{-1} |0; k^\ell\rangle \otimes |0,0\rangle$

25-dim B-field: $B_{ij}\, \alpha^i_{-1} \tilde{\alpha}^j_{-1} |0; k^\ell\rangle \otimes |0,0\rangle$

graviphoton and extra photon $\quad (\xi \cdot \alpha_{-1} \tilde{\alpha}^{25}_{-1} \pm \xi \cdot \tilde{\alpha}_{-1} \alpha^{25}_{-1}) |0; k^\ell\rangle \otimes |0,0\rangle$

(graviphoton from the 26 dim metric + another photon from the 26 dim KR field)

radion $\quad \alpha^{25}_{-1} \tilde{\alpha}^{25}_{-1} |0; k^\ell\rangle \otimes |0,0\rangle$

identified with the scalar σ

massless string spectrum $\longleftrightarrow$ massless spectrum from KK reduction of EFT

$mw \neq 0$ generically give massive states (later)

States with non-trivial m, w are obtained by acting with oscillators on the state

$$|0, k'\rangle \otimes |m, w\rangle \qquad N = \tilde{N} = 0$$

$$\text{so } mw = 0$$

$$\rightarrow \quad M^2_{(25)} = \frac{m^2}{R^2} + 4R^2 w^2 - 8$$

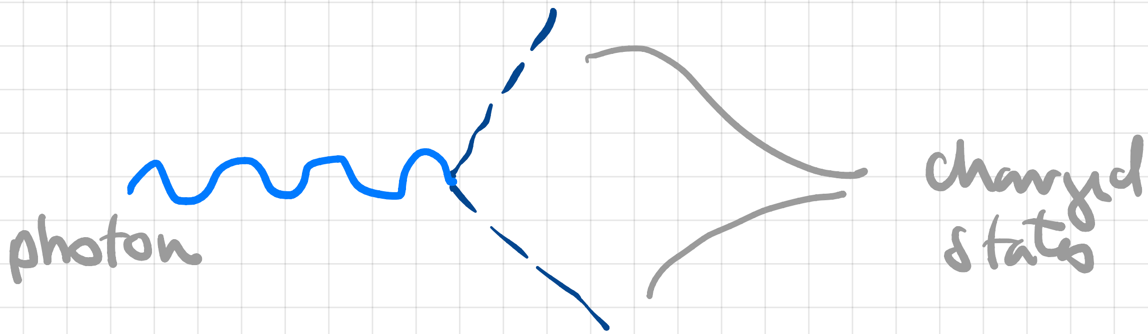
when $m = 0$ $\quad \underline{M^2_{(25)} = 4R^2 w^2 - 8}$ winding tachyon

etc.

Checking that KK-modes with $m \neq 0$ & winding modes are charged under the $\underset{A_i}{U(1)} \times \underset{\tilde{A}_i}{U(1)}$ gauge symmetries:

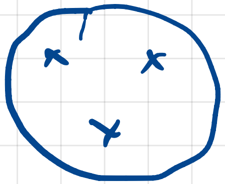
graviphoton

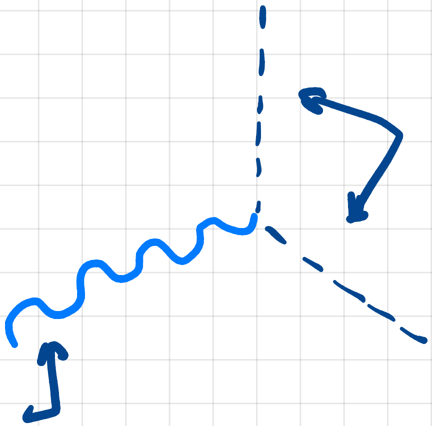
↳ compute the 3-point function in which



read off charge of the state: coefficient "in front" of the coupling of the fields

Consider then the 3-amplitude





winding tachyon

$|0,k\rangle \otimes |0,w\rangle \quad (m=0)$

$M_{25}^2 = -8 + 4R^2 w^2$

Kalb-Ramond photon

$(\xi\cdot\alpha_{-1}\tilde{\alpha}_{-1}^{25} - \xi\cdot\tilde{\alpha}_{-1}\alpha_{-1}^{25})|0,k\rangle \otimes |0,0\rangle$

Vertex operator for the KR photon

$$V_{KR}(k) \sim \int d\tau\, d\sigma\; \xi\cdot(\partial_+ X\, \partial_- X^{25} - \partial_- X\, \partial_+ X^{25})\, e^{ik\cdot X}$$

Vertex operator for the tachyon:

$$V_{m,w}(p) \sim \int d\tau\, d\sigma\; e^{ip\cdot X}\, \underbrace{e^{ip_L X^{25} + i p_R X^{25}}}_{\text{assigns } (m,w) \text{ to any state}}$$

Compute the amplitude:

$$\mathcal{A} = \underline{\langle 0, -k_3; 0, \omega |} \left(\xi \cdot \partial_+ X \partial_- X^{25} - \xi \cdot \partial_- X \partial_+ X^{25} \right) e^{ik_2 \cdot X} \underline{| 0, k_1; 0, 0 \rangle}$$

$$= \langle 0, -k_3; 0, \omega | \left(\xi \cdot \tilde{\alpha}_0 \alpha_0^{25} - \xi \cdot \alpha_0 \tilde{\alpha}_0^{25} \right) | 0, k_1 + k_2; 0, 0 \rangle$$

$$= \xi \cdot (k_1 + k_2) \langle 0, -k_3; 0, \omega | \left(\alpha_0^{25} - \tilde{\alpha}_0^{25} \right) | 0, k_1 + k_1; 0, 0 \rangle$$

$$= (2R\omega) \quad \xi \cdot k_3 \quad \delta^{(25)}(k_1 + k_2 + k_3)$$

$\longleftarrow$ winding tachyon charge under $\tilde{A}_i$ (coming from $B_{\mu\nu}$)

Similar computation for the graviphoton:

momentum $\frac{m}{R}$ is the charge under A_i

This agrees with the KK reduction

Remark: we have introduced a new scale R

In fact, we have a one parameter family of compactifications with $R \in (0, \infty)$

Or do we?

next: T-duality