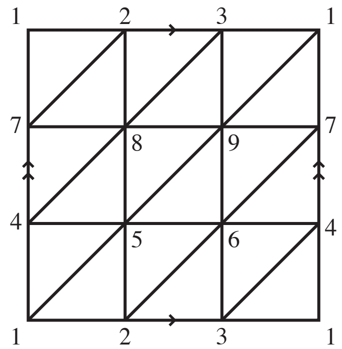
Topology & Groups

Michaelmas 2016

Question Sheet 4

1. Let K be a simplicial complex, and let α_1 and α_2 be edge paths. Suppose that α_1 and α_2 are homotopic relative to their endpoints. Show that α_1 and α_2 are equivalent as edge paths. [You should adapt the proof of Theorem III.27.]

2. Triangulate the torus as shown below



Let x and y be the loops $(1, 2, 3, 1)$ and $(1, 4, 7, 1)$, and let K be the union of these two loops (ie. K comes from the boundary of the square).

(i) Show that any edge path that starts and ends on K but with the remainder of the path missing K is equivalent to an edge path lying entirely in K.

(ii) Prove that any edge loop based at 1 is equivalent to an edge loop lying entirely in K.

(iii) Deduce that any edge loop based at 1 is equivalent to a word in the alphabet $\{x, y\}$.

(iv) Show that the edge loops xy and yx are equivalent.

(v) Deduce that any edge loop based at 1 is equivalent to $x^m y^n$, for $m, n \in \mathbb{Z}$.

(vi) Prove that if $x^m y^n \sim x^M y^N$, then $m = M$ and $n = N$. [Hint: define 'winding numbers' as in the proof of Theorem III.32.]

(vii) Deduce that the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

2. Prove that every non-trivial element of a free group has infinite order.

3. The centre $Z(G)$ of a group G is $\{g \in G : gh = hg \ \forall h \in G\}$. Let S be a set with more than one element. Prove that the centre of $F(S)$ is the identity element.

4. (i) Let F be the free group on the three generators x, y and z. For non-zero integers r, s and t, show that the subgroup of F generated by x^r, y^s and z^t is freely generated by these elements.

   (ii) Let H be the subgroup of $F(\{x, y\})$ generated by x^2, y^2, xy and yx. Show that H is not freely generated by these elements.

5. Compute an explicit free generating set for the fundamental group of the following graph: