Numerical Solution of Differential Equations II. QS 2

Question 1. Consider the system

$$L_h(y_j) = -\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + f\left(x_j, y_j, \frac{y_{j+1} - y_{j-1}}{2h}\right) = 0 \quad \text{for} \quad j = 1, 2, \ldots, n$$

with boundary values $y_0 = \alpha$ and $y_{n+1} = \beta$. This nonlinear system can be expressed in vector notation as $\Phi(y) = 0$ where the i^{th} row of the system is given by $L_h(y_j)$ above. Write Newton's method for computing an approximate solution to this system. Determine conditions on $f(\cdot, \cdot, \cdot)$ so that the linear system in Newton's method can be solved at each iteration.

Question 2. Consider the finite difference approximation

$$L_h(y_j) = -\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + f\left(x_j, \frac{y_{j-1} + y_{j+1}}{2}, \frac{y_{j+1} - y_{j-1}}{2h}\right) = 0 \quad \text{for} \quad j = 1, 2, \ldots, n$$

to $L(y) = -y'' + f(x, y, y') = 0$ with boundary values $y(x_0) = y_0 = \alpha$ and $y(x_{n+1}) = y_{n+1} = \beta$.

(a) Show that the operator has second order truncation error.

(b) Show that, under suitable conditions, L_h is stable.

(c) Combine the results from (a) and (b) to show that $\max_j |y(x_j) - y_j| = \mathcal{O}(h^2)$.

Question 3. Consider the iteration

$$y_j^{m+1} = \frac{1}{2}(y_{j-1}^m + y_{j+1}^m) - \frac{h^2}{2} f\left(x_j, \frac{y_{j-1}^m + y_{j+1}^m}{2}, \frac{y_{j+1} - y_{j-1}^m}{2h}\right) \quad \text{for} \quad j = 1, 2, \ldots, n$$

with boundary values $y_0 = \alpha$ and $y_{n+1} = \beta$. Show that the iteration converges with m. List any conditions imposed to ensure convergence.