B5.6 Nonlinear Systems

5. Global Bifurcations, Homoclinic chaos, Melnikov's method

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2018

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What we learned from Section 1&2.

- For a system of linear autonomous equations $\dot{\mathbf{x}} = A\mathbf{x}$, the solutions live on invariant spaces that can be classified according to the eigenvalues of A.
- The stable (resp. unstable, centre) linear subspace is the span of eigenvectors whose eigenvalues have a negative (resp. positive, null) real part.
- For nonlinear systems, we define the notion of asymptotic sets (α and ω limit set), the notion of attracting set, and basin of attraction.
- We define two important notions of stability for a fixed point: *(Lyapunov) stability* (i.e. "solutions remain close") and *exponential stability* i.e. ("fixed point is stable AND all nearby solutions converge to the fixed point asymptotically for long time").
- Lyapunov functions can be used to test stability. But, finding a Lyapunov function can be difficult.

What we learned from Section 3.

- For a system of autonomous equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, we are interested in the trajectories and asymptotic sets in phase space.
- At a fixed point, we can define *local stable, unstable, and centre manifolds* based on the corresponding linear subspaces of the linearised system.
- From the local stable and unstable manifolds, we can define the *global stable and unstable manifolds* by extending them to all times.
- If the unstable manifold is non-empty, the fixed point is unstable.
- If the unstable and centre manifolds are empty, the fixed point is asymptotically stable.
- If the unstable manifold is empty but the centre manifold is non-empty, we can study the dynamics on the centre manifold by *centre manifold reduction*.
- The same notions can be defined for iterative mappings and for periodic orbits.

What we learned from Section 4.

- For a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$, we are interested in bifurcations.
- Bifurcations are parameter values where a qualitative change occurs.
- For local bifurcations (at fixed points), it implies the existence of eigenvalues of the Jacobian matrix crossing the imaginary axis.
- At the bifurcation value, there is a centre manifold. Hence, we can use centre manifold techniques.
- Adding the parameter as a variable, we can define an extended centre manifold and use results of Section 3.
- We find generic behaviours of bifurcations changing the type or number of fixed points (saddle-node, transcritical, pitchfork).
- The Hopf bifurcation leads to the possibility of transforming a fixed point into a limit cycle.
- The same notions apply to mappings and periodic orbits.
- Bifurcation of maps show the new possibility of period-doubling (leading to chaos).

4. Motivation

4. Motivation

4.1 The problem

The problem

Consider the first-order system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, t) \qquad \text{where} \quad \mathbf{x} : E \subset \mathbb{R}^n, \tag{1}$$

and assume that $\mathbf{g}$ is periodic in t ($\exists\ T > 0$, s.t. $\mathbf{g}(\mathbf{x}, t + T) = \mathbf{g}(\mathbf{x}, t)$.)

Assuming, we know the dynamics of the system when $\varepsilon = 0$ and that it supports periodic and homoclinic orbits.

Problem:
What happens when $\varepsilon > 0$?
Are there still periodic orbits?
Homoclinic orbits?
New orbits?

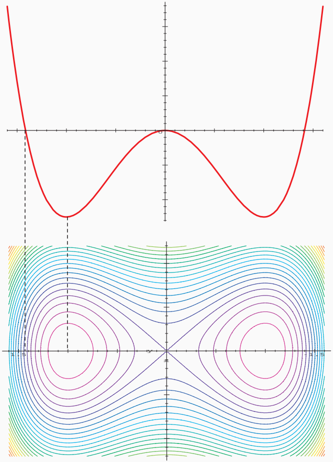
4. Motivation

4.2 A paradigm

An important example: the Duffing oscillator

$$\ddot{x} = x - x^3 + \varepsilon\left(\delta \dot{x} + \gamma \cos(t)\right) \tag{2}$$

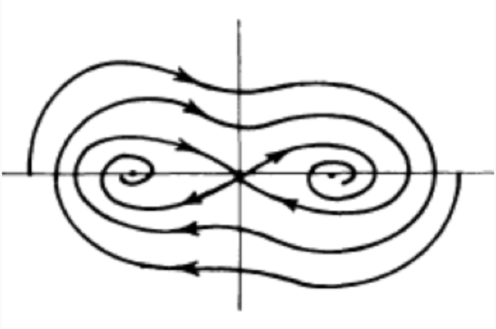
For $\varepsilon = 0$.



An important example: the Duffing oscillator

$$\ddot{x} = x - x^3 + \varepsilon\,(\delta\dot{x} + \gamma\cos(t)) \tag{3}$$

For $\varepsilon > 0, \gamma = 0, \delta > 0$.



What happens when $\varepsilon > 0, \gamma > 0, \delta > 0$? $\Longrightarrow$ CHAOS!

For what values? What does chaos mean?

Main idea

Our construction will be in four steps of increasing complexity

Step 1: Bernoulli shift (the simplest dynamical system with chaos)

Step 2: Smale's horseshoe (a geometric construction)

Step 3: Homoclinic chaos in ODEs

Step 4: Melnikov's method (an explicit method to detect chaos)

4. Motivation

4.3 Step 1: Bernoulli Shift

A simple dynamical system

To define a dynamical system we need:
- A *phase space* Σ.
- The *dynamics* on Σ (how elements of Σ are mapped to other elements).

1. The phase space.
For the Bernoulli shift we define Σ as the set of bi-infinite sequence of 0 and 1:

$$s \in \Sigma : \qquad s = \{\ldots, s_{-n}, \ldots, s_{-1} | s_0, s_1, \ldots, s_n, \ldots\}, \tag{4}$$

where s_i is equal to 0 or 1.

Bernoulli Shift

$s \in \Sigma$:

$$s = \{\ldots, s_{-n}, \ldots, s_{-1} | s_0, s_1, \ldots, s_n, \ldots\}, \tag{5}$$

where s_i is equal to 0 or 1.

Notion of *distance* on Σ. Take two elements $s, s' \in \Sigma$:

$$d(s, s') = \sum_{i \in \mathbb{Z}} \frac{|s_i - s'_i|}{2^{|i|}} \tag{6}$$

Tow elements are close if their central blocks agree,

Bernoulli Shift

2. The dynamics on Σ

Define the shift map $\sigma : \Sigma \mapsto \Sigma$. If

$$s = \{\ldots, s_{-n}, \ldots, s_{-2}, s_{-1} | s_0, s_1, \ldots, s_n, \ldots\}, \tag{7}$$

then

$$\sigma(s) = \{\ldots, s_{-n}, \ldots, s_{-1}, s_0 |, s_1, \ldots, s_n, \ldots\}, \tag{8}$$

Equivalently

$$(\sigma(s))_i = s_{i+1}. \tag{9}$$

Possible orbits?

Bernoulli shift

Theorem 4.1

The shift map has:

1. *a countable infinity of periodic orbits with arbitrary periods;*
2. *an uncountable infinity of non-periodic orbits;*
3. *a dense orbit.*

What is a dense orbit?

Definition 4.2

A *dense orbit* for the shift map is a particular orbit $s_d \in \Sigma$ such that for any $s \in \Sigma$ and $\epsilon > 0$, $\exists n \in \mathbb{N}$ such that $d(\sigma^n(s_d), s) < \epsilon$.

Bernoulli shift

Theorem 4.3

The shift map has:

1. *a countable infinity of periodic orbits with arbitrary periods;*
2. *an uncountable infinity of non-periodic orbits;*
3. *a dense orbit.*

Proof:

Sensitive dependence to initial conditions

Two important notions in dynamical systems.
Let Λ be an invariant compact set for an invertible iterative map $f : \mathcal{M} \to \mathcal{M}$.

Definition 4.4

f has *sensitivity to initial conditions* on Λ if $\exists \epsilon > 0$ such that for any $p \in \Lambda$ and any neighbourhood U of p, there exists $p' \in U$ and $n \in \mathbb{N}$ such that $|f^n(p) - f^n(p')| > \epsilon$.

Definition 4.5

f is *topologically transitive* on Λ if for any open sets $U, V \in \Lambda$ $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

Sensitive dependence to initial conditions

Together they lead to the notion of chaos:

Definition 4.6

Let Λ be an invariant compact set for an invertible iterative map $f : \mathcal{M} \to \mathcal{M}$. Then f is *chaotic* on Λ if it has sensitivity to initial conditions on Λ **and** is topologically transitive on Λ.

Theorem 4.7

The shift map is chaotic on Σ.

4. Motivation

4.4 Step 2: Smale's horseshoe

Smale’s horseshoe

The construction of Smale’s horseshoe is given in the file
B56-Section5b.pdf