Normal Form Transformations for Hopf Bifurcations

Consider the 2D system $(\dot{x}, \dot{y}) = (p(x, y), q(x, y))$. Suppose the linearisation at a fixed point (x_0, y_0) shows that a pair of complex eigenvalues cross the imaginary axis $\mathrm{Re}(\lambda) = 0$ at a bifurcation point $\tilde{\mu} = \tilde{\mu}_0$. In order to put the system into the normal form for a Hopf bifurcation, i.e.

$$\begin{aligned} \dot{r} &= \mu r - ar^3 + O(r^5), \\ \dot{\theta} &= \omega - br^2 + O(r^4), \\ \text{or, equivalently} \quad \dot{z} &= (\mu + \mathrm{i}\omega)z - (a + \mathrm{i}b)|z|^2 z + O(z^5), \end{aligned}$$

it is in general necessary to do the following (see Glendinning pp227–243):

1. Shift coordinates by writing $(\tilde{x}, \tilde{y}) = (x - x_0(\tilde{\mu}), y - y_0(\tilde{\mu}))$ and $\mu = \tilde{\mu} - \tilde{\mu}_0$ so that the fixed point is at the origin $\tilde{x} = \tilde{y} = 0$ for all μ and the bifurcation is at $\mu = 0$.

2. If necessary, rescale μ so that the eigenvalues are $\lambda = \mu \pm \mathrm{i}\omega$. Make a linear change of basis so that the Jacobian is in the Jordan Normal Form
$$\begin{pmatrix} \mathrm{Re}(\lambda) & -\mathrm{Im}(\lambda) \\ \mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix}.$$

3. Drop the ˜s and write (x, y) as the single complex-valued variable $z = x + \mathrm{i}y$. Set $\mu = 0$. Then, Taylor expanding the right-hand-sides we see that the ODEs take the form $\dot{z} = \mathrm{i}\omega z + a_1 z^2 + a_2 z\overline{z} + a_3 \overline{z}^2 + O(3)$. It turns out that **all** these quadratic terms can be removed by making a suitable choice of the coefficients α_i in a near-identity change of coordinates $z = w + \alpha_1 w^2 + \alpha_2 w\overline{w} + \alpha_3 \overline{w}^2$. [Note: the algebra can be done by differentiating the inverse $w = z - \alpha_1 z^2 - \alpha_2 z\overline{z} - \alpha_3 \overline{z}^2 + O(3)$ and then substituting for $\dot{z}$ and $\dot{\overline{z}}$.]

4. Now we may attempt to eliminate all the cubic terms in $\dot{w} = \mathrm{i}\omega z + b_1 w^3 + b_2 w^2\overline{w} + b_3 w\overline{w}^2 + b_4\overline{w}^3 + O(4)$ by a suitable choice of the coefficients β_i in another near-identity map $w = Z + \beta_1 Z^3 + \beta_2 Z^2\overline{Z} + \beta_3 Z\overline{Z}^2 + \beta_4\overline{Z}^3$. It turns out that the $b_2 w^2\overline{w}$ term **cannot** be eliminated!

5. Continuing with more near-identity transformations, it is possible to eliminate all the quartic terms to show that the next term remaining in the normal form is quintic.

The chief point of these steps is to find the sign of a in the normal form, and hence decide whether the bifurcation is subcritical or supercritical.

If steps 1 and 2 have already been done so that the system is in the form

$$\begin{aligned} \dot{x} &= \mu x - \omega y + f(x, y) \\ \dot{y} &= \omega x + \mu y + g(x, y) \end{aligned}$$

then

$$a = \frac{-1}{16\omega}\Big\{(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy})\omega + f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\Big\}.$$