# B5.6 Nonlinear Systems

## 2. Nonlinear systems

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## What we learned from Section 1.

- Solutions of linear autonomous system $\dot{\mathbf{x}} = A\mathbf{x}$ are given by the matrix exponential $e^{tA}$.
- The matrix exponential defines a linear flow (mapping sets to sets in $\mathbb{R}^n$).
- The set of eigenvalues of $A$ can be used to define three subspaces based on their real part.
- The stable (resp. unstable, centre) linear subspace is the span of eigenvectors whose eigenvalues have negative (resp. positive, null) real parts.
- These linear subspaces are invariant spaces (initial conditions in an invariant set define solutions that remain for all time within that set).
- The solution of linear maps can be classified in a similar way.

# 2. Nonlinear systems

# 2. Nonlinear systems

## 2.1 Existence and uniqueness

# The problem

Consider the nonlinear, autonomous, first-order system of differential equations:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{1}$$

where $\mathbf{f} : E \subset \mathbb{R}^n \to \mathbb{R}^n$ is the *vector field*.
In general, this equation cannot be solved.
**Problems**:
What are the possible solutions (from a geometric point of view)?
What is the stability of such solutions (behaviour of nearby solutions)?

# Existence theorem

**Theorem 2.1**

*Let $f \in C^1(E)$ and $\mathbf{x}_0 \in E$, then there exists $c > 0$ s.t.*

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{2}$$

*has one and only one solution $\mathbf{x}(t)$ on $[-c, c]$.*

NB1: **For the rest of this course, unless otherwise specified, we will assume that the maximum interval of existence is $\mathbb{R}$** (we are interested in global behaviour).

NB2: The general conditions guaranteeing the existence of global solutions are not obvious.

# 2. Nonlinear systems

## 2.2 Flows, asymptotic sets, and invariant sets

# Flows

The space $E \subset \mathbb{R}^n$ on which the solutions live is called the *phase space*. We assume that the maximum interval of existence is $\mathbb{R}$ (i.e. solutions are defined for all time for all initial conditions).
An *orbit* based on $\mathbf{x}_0$ is the curve $\Gamma_{\mathbf{x}_0} \subset E$ defined by

$$\Gamma_{\mathbf{x}_0} = \{\mathbf{x}(t) \in \mathbb{R}^n; t \in \mathbb{R}, \mathbf{x}(t_0) = \mathbf{x}_0, \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\} \tag{3}$$

The *flow* is the map $\varphi_t : E \to E$ such that

$$\varphi_t(\mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}_0) \qquad \forall\ \mathbf{x}_0 \in K \subset E \tag{4}$$

# Flows

**Properties of flows:**

- $\varphi_0 = \mathbf{1}$ (the identity map)
- $\varphi_{t+s}(\mathbf{x}) = \varphi_t(\varphi_s(\mathbf{x})) = \varphi_s(\varphi_t(\mathbf{x})), \quad \forall\ \mathbf{x} \in \mathbb{R}^n$
- Let $U$ be a neighborhood of $\mathbf{x}_0$ and $V = \varphi_t(U)$, then

$$\varphi_{-t}(\varphi_t(\mathbf{x})) = \mathbf{x}, \qquad \forall\ \mathbf{x} \in U \tag{5}$$

$$\varphi_t(\varphi_{-t}(\mathbf{y})) = \mathbf{y}, \qquad \forall\ \mathbf{y} \in V \tag{6}$$

**Definition 2.2**

Consider a vector field $\mathbf{f} \in C^1(E)$, defining a flow $\varphi_t : E \to E$.
Then $S \subset E$ is an *invariant set* of such that $\varphi_t$ if

$$\varphi_t(S) \subset S \quad \forall\ t \in \mathbb{R}. \tag{7}$$

**Example:**

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y + x^2 \end{cases} \tag{8}$$

**Definition 2.3**

A point $\mathbf{p} \in E$ is an *$\omega$-limit point* of $\varphi_t$ if there exists a sequence of time $t_1 < t_2 < \ldots < t_n$, with $t_i \to \infty$ as $i \to \infty$ such that

$$\lim_{i \to \infty} \varphi_{t_i}(\mathbf{x}) = \mathbf{p}. \tag{9}$$

Similarly, a point $\mathbf{p} \in E$ is an *$\alpha$-limit point* of $\varphi_t$ if there exists a sequence of time $t_1 > t_2 > \ldots > t_n$, with $t_i \to -\infty$ as $i \to \infty$ such that

$$\lim_{i \to -\infty} \varphi_{t_i}(\mathbf{x}) = \mathbf{p}. \tag{10}$$

# Asymptotic sets

**Example:**

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases} \tag{11}$$

# Attracting sets

**Definition 2.4**

An attracting set is a closed invariant set $A$ such that there exists a neighborhood $U$ of $A$ with the properties

$$\begin{cases} \varphi_t(\mathbf{x}) \in U & \forall\ t \geq 0 \\ \varphi_t(\mathbf{x}) \xrightarrow[t \to \infty]{} A & \forall\ \mathbf{x} \in U \end{cases} \tag{12}$$

# Domain of attraction

For a given attracting set $A$, with neighborhood $U$ as above, the *domain of attraction* is the set of all initial conditions that have $A$ as $\omega$-limit set. That is

$$D(A) = \bigcup_{t \leq 0} \varphi_t(U) \tag{13}$$

# Attracting sets

**Example:**

$$\dot{x} = \begin{cases} -x^4 \sin(\pi/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{14}$$

# 2. Nonlinear systems

## 2.3 Stability

# Stability

We consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E \subset \mathbb{R}^n$, $f \in C^1(E)$.
The simplest solutions are fixed points.

A *fixed point* $\mathbf{x}_0$ is a constant solution of the system, that is $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.

We want a definition of the intuitive idea of stability:
"solutions close to a given invariant set remains close to that set for all time."

# Lyapunov stability

Let $B_r(\mathbf{x})$ denote a closed ball of radius $r$ around $\mathbf{x}$ (the set of points with a distance less than $r$ from $\mathbf{x}$).

**Definition 2.5**

The fixed point $\mathbf{x}_0$ is *(Lyapunov) stable* if $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that $\forall\ \mathbf{x} \in B_\delta(\mathbf{x}_0)$ and $t \geq 0$, we have $\varphi_t(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0)$.

# Lyapunov stability

**Example:**

$$\begin{cases} \dot{x} = y \\ \dot{y} = -4x \end{cases} \tag{15}$$

# Asymptotic stability

A stronger notion of stability:

**Definition 2.6**

The fixed point $\mathbf{x}_0$ is *asymptotically stable* if it is Lyapunov stable and $\exists\ \delta > 0$ such that $\forall\ \mathbf{x} \in B_\delta(\mathbf{x}_0)$ we have $\varphi_t(\mathbf{x}) \xrightarrow[t\to\infty]{} \mathbf{x}_0$.

**Example:**

$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2 \frac{\theta}{2} \end{cases} \tag{16}$$

# Asymptotic stability

For linear systems $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}_0 = \mathbf{0}$ is always a fixed point.

If $A$ is a semi-simple matrix, we have

- $\mathbf{x} = \mathbf{0}$ is asymptotically stable if $\text{Re}(\lambda) < 0, \quad \forall\, \lambda \in \text{Spec}(A)$.
- $\mathbf{x} = \mathbf{0}$ is stable if $\text{Re}(\lambda) \leq 0, \quad \forall\, \lambda \in \text{Spec}(A)$.

NB: The first property remains true in the general case, but not the second one (can you find a counter-example?).

# 2. Nonlinear systems

## 2.4 Lyapunov functions

# Lyapunov functions

We consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E \subset \mathbb{R}^n$, $\mathbf{f} \in C^1(E)$.
Assume that this system has a fixed point $\mathbf{x}_0$ (so that $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$).

**Definition 2.7**

A function $V : W \subset E \to R$, $V \in C^1(W)$ is a *Lyapunov function* if there exists a neighborhood $W$ of $\mathbf{x}_0$ on which it satisfies

1. $V(\mathbf{x}_0) = 0$, and $V(\mathbf{x}) > 0 \; \forall \; \mathbf{x} \in W \setminus \{x_0\}$.
2. $\dot{V}(\mathbf{x}) \leq 0 \; \forall \; \mathbf{x} \in W \setminus \{x_0\}$.

# Lyapunov functions

If a Lyapunov function is known, the following stability results hold:

**Theorem 2.8**

1. *If $V$ is Lyapunov function of* $\mathbf{f}$ *in a neighborhood of* $\mathbf{x}_0$, *then* $\mathbf{x}_0$ *is stable.*
2. *If, in addition,* $\dot{V}(\mathbf{x}) < 0 \; \forall \; \mathbf{x} \in W \setminus \{x_0\}$, *then* $\mathbf{x}_0$ *is asymptotically stable.*

**Proof** (Perko, p.131):

# Lyapunov functions

**Example:** The damped nonlinear spring

$$m\ddot{x} + k(x + x^3) + \alpha\dot{x} = 0, \qquad \alpha > 0. \tag{17}$$

# Extra reading material

Extra material from the following books


[S ] Strogatz, *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering* (Westview Press, 2000).

[D ] Drazin, *Nonlinear Systems* (Cambridge University Press, Cambridge, 1992).

[P ] Perko, *Differential Equations and Dynamical Systems* (Second edition, Springer, 1996).


2 Nonlinear Systems

- 2.1 Fundamental theorems. [P70]
- 2.2 Flow, asymptotic behaviour. [P87,95]
- 2.3 Stability theory, Lyapunov function [P129]