Prelims Mathematics 2021-22

June 21, 2021

Contents

1 Foreword

Synopses

The synopses give some additional detail and show how the material is split between the different lecture courses. They include details of recommended reading.

2 Syllabus

The syllabus here is that referred to in the Examination Regulations 2021 Special Regulations for the Preliminary Examination in Mathematics & Philosophy (https://www.admin.ox.ac.uk/examregs/Lin to an external site.). Examination Conventions can be found at: http://www.maths.ox.ac.uk/members/students/ courses/examina...Links to an external site..

Mathematics I

The natural numbers and their ordering. Induction as a method of proof, including a proof of the binomial theorem with non-negative integral coefficients.

Sets. Examples including $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and intervals in $\mathbb{R}$. Inclusion, union, intersection, power set, ordered pairs and cartesian product of sets. Relations. Definition of an equivalence relation. Examples.

Functions: composition, restriction; injective (one-to-one), surjective (onto) and invertible functions; images and preimages.

Systems of linear equations. Matrices and the beginnings of matrix algebra. Use of matrices to describe systems of linear equations. Elementary Row Operations (EROs) on matrices. Reduction of matrices to echelon form. Application to the solution of systems of linear equations.

Inverse of a square matrix. Reduced row echelon (RRE) form and the use of EROs to compute inverses; computational efficiency of the method. Transpose of a matrix; orthogonal matrices.

Vector spaces: definition of a vector space over a field (such as $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). Subspaces. Many explicit examples of vector spaces and subspaces.

Span of a set of vectors. Examples such as row space and column space of a matrix. Linear dependence and independence. Bases of vector spaces; examples. The Steinitz Exchange Lemma; dimension. Application to matrices: row space and column space, row rank and column rank. Coordinates associated with a basis of a vector space.

Use of EROs to find bases of subspaces. Sums and intersections of subspaces; the dimension formula. Direct sums of subspaces.

Linear transformations: definition and examples (including projections associated with direct-sum decompositions). Some algebra of linear transformations; inverses. Kernel and image, Rank-Nullity Theorem. Applications including algebraic characterisation of projections (as idempotent linear transformations).

Matrix of a linear transformation with respect to bases. Change of Bases Theorem. Applications including proof that row rank and column rank of a matrix are equal.

Bilinear forms; real inner product spaces; examples. Mention of complex inner product spaces. Cauchy–Schwarz inequality. Distance and angle. The importance of orthogonal matrices.

Introduction to determinant of a square matrix: existence and uniqueness. Proof of existence by induction. Proof of uniqueness by deriving explicit formula from the properties of the determinant. Permutation matrices. (No general discussion of permutations). Basic properties of determinant, relation to volume. Multiplicativity of the determinant, computation by row operations.

Determinants and linear transformations: definition of the determinant of a linear transformation, multiplicativity, invertibility and the determinant.

Eigenvectors and eigenvalues, the characteristic polynomial, trace. Eigenvectors for distinct eigenvalues are linearly independent. Discussion of diagonalisation. Examples. Eigenspaces, geometric and algebraic multiplicity of eigenvalues. Eigenspaces form a direct sum.

Gram-Schmidt procedure. Spectral theorem for real symmetric matrices. Quadratic forms and real symmetric matrices. Application of the spectral theorem to putting quadrics into normal form by orthogonal transformations and translations. Statement of classification of orthogonal transformations.

Axioms for a group and for an Abelian group. Examples including geometric symmetry groups, matrix groups (GL_n, SL_n, O_n, U_n), cyclic groups. Products of groups.

Permutations of a finite set under composition. Cycles and cycle notation. Order. Transpositions; every permutation may be expressed as a product of transpositions. The parity of a permutation is well-defined via determinants. Conjugacy in permutation groups.

Subgroups; examples. Intersections. The subgroup generated by a subset of a group. A subgroup of a cyclic group is cyclic. Connection with hcf and lcm. Bezout's Lemma.

Recap on equivalence relations including congruence mod n and conjugacy in a group. Proof that equivalence classes partition a set. Cosets and Lagrange's Theorem; examples. The order of an element. Fermat's Little Theorem.

Isomorphisms, examples. Groups of order 8 or less up to isomorphism (stated without proof). Homomorphisms of groups with motivating examples. Kernels. Images. Normal subgroups. Quotient groups; examples. First Isomorphism Theorem. Simple examples determining all homomorphisms between groups.

Group actions; examples. Definition of orbits and stabilizers. Transitivity. Orbits partition the set. Stabilizers are subgroups.

Orbit-stabilizer Theorem. Examples and applications including Cauchy's Theorem and to conjugacy classes.

Orbit-counting formula. Examples.

The representation $G \to \mathrm{Sym}(S)$ associated with an action of G on S. Cayley's Theorem. Symmetry groups of the tetrahedron and cube.

Mathematics II

Complex numbers and their arithmetic. The Argand diagram (complex plane). Modulus and argument of a complex number. Simple transformations of the complex plane. De Moivre's Theorem; roots of unity. Euler's theorem; polar form $re^{\mathrm{i}\theta}$ of a complex number. Polynomials and a statement of the Fundamental Theorem of Algebra.

Real numbers: arithmetic, ordering, suprema, infima; the real numbers as a complete ordered field. Definition of a countable set. The countability of the rational numbers. The reals are uncountable. The complex number system. The triangle inequality.

Sequences of real or complex numbers. Definition of a limit of a sequence of numbers. Limits and inequalities. The algebra of limits. Order notation: O, o.

Subsequences; a proof that every subsequence of a convergent sequence converges to the same limit; bounded monotone sequences converge. Bolzano–Weierstrass Theorem. Cauchy's convergence criterion.

Series of real or complex numbers. Convergence of series. Simple examples to include geometric progressions and some power series. Absolute convergence, Comparison Test, Ratio Test, Integral Test. Alternating Series Test.

Power series, radius of convergence. Examples to include definition of and relationships between exponential, trigonometric functions and hyperbolic functions.

Definition of the function limit. Definition of continuity of functions on subsets of $\mathbb{R}$ and $\mathbb{C}$ in terms of ε and δ. Continuity of real valued functions of several variables. The algebra of continuous functions; examples, including polynomials. Intermediate Value Theorem for continuous functions on intervals. Boundedness, maxima, minima and uniform continuity for continuous functions on closed intervals. Monotone functions on intervals and the Inverse Function Theorem.

Sequences and series of functions, uniform convergence. Weierstrass's M-test for uniformly convergent series of functions. Uniform limit of a sequence of continuous functions is continuous. Continuity of functions defined by power series.

Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. The derivative of a function defined by a power series is given by the derived series (proof not examinable). Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem, Mean Value Theorem, and Cauchy's (Generalized) Mean Value Theorem with applications: Constancy Theorem, monotone functions, exponential function and trigonometric functions. L'Hôpital's Formula. Taylor's Theorem with remainder in Lagrange's form; examples. The binomial expansion with arbitrary index.

Step functions, their integral, basic properties. Minorants and majorants of bounded functions on bounded intervals. Definition of Riemann integral. Elementary properties of Riemann integrals: positivity, linearity, subdivision of the interval.

The application of uniform continuity to show that continuous functions are Riemann integrable on closed bounded intervals; bounded continuous functions are Riemann integrable on bounded intervals.

The Mean Value Theorem for Integrals. The Fundamental theorem of Calculus; integration by parts and by substitution.

The interchange of integral and limit for a uniform limit of integrable functions on a bounded interval. Term-by-term integration and differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly).

Mathematics IIIP

General linear homogeneous ODEs: integrating factor for first order linear ODEs, second solution when one solution is known for second order linear ODEs. First and second order linear ODEs with constant coefficients. General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation. Simple examples of finding particular integrals by guesswork.

Introduction to partial derivatives. Second order derivatives and statement of condition for equality of mixed partial derivatives. Chain rule, change of variable, including planar polar coordinates. Solving some simple partial differential equations (e.g. $f_{xy} = 0$, $f_x = f_y$).

Parametric representation of curves, tangents. Arc length. Line integrals.

Jacobians with examples including plane polar coordinates. Some simple double integrals calculating area and also $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\mathrm{d}A$.

Simple examples of surfaces, especially as level sets. Gradient vector; normal to surface; directional derivative; $\int_A^B \nabla\phi \cdot \mathrm{d}\mathbf{r} = \phi(B) - \phi(A)$.

Taylor's Theorem for a function of two variables (statement only). Critical points and classification using directional derivatives and Taylor's theorem. Informal (geometrical) treatment of Lagrange multipliers.

Sample space, events, probability measure. Permutations and combinations, sampling with or without replacement. Conditional probability, partitions of the sample space, law of total probability, Bayes' Theorem. Independence.

Discrete random variables, probability mass functions, examples: Bernoulli, binomial, Poisson, geometric. Expectation, expectation of a function of a discrete random variable, variance. Joint distributions of several discrete random variables. Marginal and conditional distributions. Independence. Conditional expectation, law of total probability for expectations. Expectations of functions of more than one discrete random variable, covariance, variance of a sum of dependent discrete random variables.

Solution of first and second order linear difference equations. Random walks (finite state space only).

Probability generating functions, use in calculating expectations. Examples including random sums and branching processes.

Continuous random variables, cumulative distribution functions, probability density functions, examples: uniform, exponential, gamma, normal. Expectation, expectation of a function of a continuous random variable, variance. Distribution of a function of a single continuous random variable. Joint probability density functions of several continuous random variables (rectangular regions only). Marginal distributions. Independence. Expectations of functions of jointly continuous random variables, covariance, variance of a sum of dependent jointly continuous random variables.

Random sample, sums of independent random variables. Markov's inequality, Chebyshev's inequality, Weak Law of Large Numbers.