Numerical Linear Algebra. QS 2 *(MT 2017)*

Optional questions: Qns 5, 8

1. For the linear least squares problem: $\min_x \|A\,x - b\|_2$, show that the solution obtained by QR factorisation of A is the same as the vector y obtained by solving the (square) linear system of equations $A^{\mathrm{T}} A\,y = A^{\mathrm{T}} b$ (the "normal equations").

Can you devise a way of solving the linear least squares problem by employing the SVD?

2. For given positive data $\{y_i\}$ at times $\{t_i\}$ respectively, formulate a linear least squares problem to determine the best values of the three parameters a, λ, μ in a model of the form

$$y = a e^{\lambda(t-\mu)}.$$

Comment on this least squares problem. If the data is $\{12, 5, 2, 1\}$ at $\{0, 1, 2, 3\}$ use matlab to compute the QR factorization (`[Q,R]=qr(A)`) and thence compute the solution.

3. Show that if "shifts" are incorporated into the QR algorithm so that it becomes

$$\begin{cases} \text{Factor} \quad (A_k - \mu_k\, I) = Q_k\, R_k \\ \text{and} \qquad A_{k+1} = R_k\, Q_k + \mu_k\, I \end{cases}$$

for $k = 1, 2, \cdots$ with $A = A_1$, then all the matrices A_k, $k = 1, 2, \cdots$ are similar and thus have the same eigenvalues.

4. In matlab, by typing

$$\begin{cases} [q, r] = qr(a) \qquad \text{(see help qr)} \\ a = r * q, \end{cases}$$

successively, observe convergence of the QR algorithm for the matrix

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \qquad (a = [9, 1, 1; 1, 5, 0; 1, 1, 2]).$$

Use the matlab command **eig** to check.

5. By showing that for distinct $u, v \in \mathbb{R}^n$ with $\|u\|_2 = \|v\|_2$ there exists w such that $u^{\mathrm{T}} H(w) = v^{\mathrm{T}}$ where H is the Householder matrix, constructively prove that for any $A \in \mathbb{R}^{n\times n}$ there exists an orthogonal matrix Q and a lower triangular matrix L such that $A = L\,Q$.

6. If $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ show that the choice of rotation angle $\theta = \cos^{-1}\left(\frac{x_i}{\sqrt{x_i^2+x_k^2}}\right)$ makes $y_k = 0$ where $y = J(i,k)\,x$. Hence show that a QR factorisation of a tridiagonal (or more generally an upper Hessenberg) matrix $A \in \mathbb{R}^{n\times n}$ can be achieved using Givens rotations as follows:

$$J(n-1,n)\cdots J(2,3)\,J(1,2)\,A = R.$$

What is Q?

7. Perform Gauss Elimination by hand on the linear system $A\,x = [11, 6, 4]^{\mathrm{T}}$ where A is the matrix in Question 4. Check your multipliers and the resulting upper triangular matrix by doing $[L, U] = lu(A)$ in matlab.

8. Note that no row swaps would have occured in the above example even if partial pivoting was employed (as it is in matlab's lu). Change the (3,1) entry in A to 10 so that pivoting (row swapping) occurs at least in the first column: do $[L, U] = lu(A)$ and check that $P\,A = L\,U$.

9. $A \in \mathbb{R}^{n\times n}$ is "Strictly Column Diagonally Dominant", (SCDD) if

$$|a_{jj}| > \sum_{i=1, i\neq j}^{n} |a_{ij}|$$

for each $j = 1, 2, \cdots, n$.

If A is SCDD and a partial pivoting strategy is used with Gauss Elimination, why is there no row swapping at the first stage (i.e. when zeros are introduced into the first column)? If after this stage the matrix is in the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & \boxed{\quad B \quad} & \\ 0 & & & \end{pmatrix},$$

by considering the general column of B

$$\left(a_{2j} - \frac{a_{21}}{a_{11}}\,a_{1j}, a_{3j} - \frac{a_{31}}{a_{11}}\,a_{1j}, \cdots, a_{nj} - \frac{a_{n1}}{a_{11}}\,a_{1j}\right)^{\mathrm{T}}$$

prove that B is SCDD. Hence by induction row swapping is not required at any stage for SCDD matrices.

10. If $A\,x = b$ and $(A + \delta\,A)\,(x + \delta\,x) = b$ show that

$$\frac{\|\delta\,x\|}{\|x + \delta\,x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta\,A\|}{\|A\|}.$$

11. (Optional: use for revision later if you wish)

(Part C exam question from 2009) What is an orthogonal matrix? Suppose that $Q \in \mathbb{R}^{m\times m}$ and $Z \in \mathbb{R}^{n\times n}$ are orthogonal matrices and define $Q_\ell \in \mathbb{R}^{m\times \ell}$ to be the matrix comprising the first ℓ columns of Q, analogously $Z_h \in \mathbb{R}^{n\times h}$ to be the matrix comprising the first h columns of Z. If $A \in \mathbb{R}^{m\times n}$, $m \geq n$, is an arbitrary matrix, prove that $\|AZ_h\|_2 \leq \|A\|_2$ for any integer h with $1 \leq h \leq n$. For what $\ell, 1 \leq \ell \leq m$ is it necessarily true that $\|Q_\ell^T A\|_2 = \|A\|_2$?

What is a Givens rotation matrix, $J(i,k,\theta), i \neq k$? Show that $J(i,k,\theta)$ is orthogonal and that if $y = J(i,k,\theta)x$ then a particular choice of θ which you should identify will ensure that $y_k = 0$ where y_j is the j^{th} entry of the vector y. What value of θ would ensure that $y_i = 0$?

Using these results, what sequence of Givens rotation matrices will perform a QR factorisation of a tridiagonal matrix $A \in \mathbb{R}^{n\times n}$? What sequence of Givens rotation matrices will perform a QR factorisation of a general full matrix $A \in \mathbb{R}^{n\times n}$?