Problem Sheet 1

1. By taking the scalar product with an arbitrary constant vector $\mathbf{k}$, use Stokes' Theorem to show that
$$\left[\int_C \mathbf{f}\wedge\mathbf{t}\,\mathrm{d}s\right]_i = \int_S \left[n_i(\nabla\cdot\mathbf{f}) - n_j\partial_i f_j\right]\,\mathrm{d}S,$$
for any vector function $\mathbf{f}$, where $\mathbf{t}$ is the unit tangent to a curve C (which bounds the surface S), the vector $\mathbf{n}$ is the unit normal to S and we use the summation convention.

   Let $\boldsymbol{\nu} = \mathbf{t}\wedge\mathbf{n}$ and choose $\mathbf{f} = \gamma\mathbf{n}$, defined off the surface by the extension that is independent of position in the normal direction $\mathbf{n}$. Show that
$$\int_C \gamma\boldsymbol{\nu}\,\mathrm{d}s = \int_S \left[\nabla\gamma - \gamma(\nabla\cdot\mathbf{n})\mathbf{n}\right]\,\mathrm{d}S.$$

   [You may need the standard identities
$$(\mathbf{f}\wedge\mathbf{k})\cdot\mathbf{t} = -\mathbf{k}\cdot(\mathbf{f}\wedge\mathbf{t}),$$
$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$
   ]

2. Recall the Laplace–Young equation governing the shape $z = h(x)$ of a meniscus near a vertical, planar wall
$$\ell_c^2\frac{h_{xx}}{(1+h_x^2)^{3/2}} = h.$$
   Integrating the full Laplace–Young equation once (without making an assumption of shallow slopes), show that the rise height of the meniscus on the wall, h_0, is given in terms of the contact angle of the liquid θ by
$$h_0 = \pm\ell_c\left[2(1-\sin\theta)\right]^{1/2}$$
   and discuss when each of the $\pm$ branches of the result are appropriate.

   Show that the total area displaced by the meniscus is given by
$$A = \int_0^\infty h\ \mathrm{d}x = \ell_c^2\cos\theta$$
   and comment on this result in the light of the generalized Archimedes' principle discussed in lecture 3.

3. With r, ϕ, z denoting cylindrical polar coordinates, fluid occupies the domain

$$\left\{(r,\phi,z)\ \middle|\ 0<r<\infty, \quad \phi\in[-\alpha,\alpha], \quad z\in[0,h(r,\phi)]\right\}$$

where $h(r,\phi)>0$ denotes the height of the free surface above the (x,y) plane.

The boundary at $z=0$ is rigid as are the boundaries at $\phi=\pm\alpha$, while the boundary at $z=h(r,\phi)$ is a static free surface, with contact lines at $z=h(r,\pm\alpha)$.

Sketch the domain and show that the linearised Laplace–Young equation enforces the equation

$$h=\ell_c^2\,\nabla^2 h,$$

for the region

$$\left\{(r,\phi)\ \middle|\ 0<r<\infty, \quad \phi\in[-\alpha,\alpha]\right\},$$

where the constant ℓ_c is to be determined.

Show that the boundary conditions are

$$\frac{1}{r}h_\phi(\alpha)=\cot\theta, \qquad \frac{1}{r}h_\phi(-\alpha)=-\cot\theta,$$

where θ is the contact angle between the air-fluid interface and the rigid boundary at $\phi=\pm\alpha$.

Explain why one must assume that $|\pi/2-\theta|\ll 1$ for self consistency.

When $\alpha=\pi/4$ show that the unique solution is

$$h=\ell_c\cot\theta\left[e^{-r\sin(\pi/4-\phi)/\ell_c}+e^{-r\sin(\phi+\pi/4)/\ell_c}\right].$$

4. Consider a static blob of fluid lying underneath a horizontal plate, $z=0$, and occupying the region

$$\left\{(x,z)\middle|-x_0\le x\le x_0, \quad h(x)\le z\le 0\right\}.$$

The liquid has a contact angle $\theta\ll 1$. Assuming that $|h_x|\ll 1$, show that h satisfies

$$\ell_c^2 h_{xxx}+h_x=0.$$

where ℓ_c is a constant, to be determined. Show that

$$h(x)=\theta\ell_c\left[\cot(x_0/\ell_c)-\frac{\cos(x/\ell_c)}{\sin(x_0/\ell_c)}\right],$$

and confirm that the assumption $|h_x|\ll 1$ is justified under certain circumstances, which you should state. Show that the cross sectional area of the drop, A, is given by

$$\frac{A}{2} = \theta\ell_c^2\left(1-\frac{x_0}{\ell_c}\cot(x_0/\ell_c)\right). \qquad (1)$$

Is there a maximum cross sectional area of fluid which can hang beneath the plate in this way? Does this accord with your physical intuition? If not, why not?

Comments and corrections to Andreas Münch
`http://people.maths.ox.ac.uk/muench/`