STRING THEORY I

Lecture 15

$\boxed{6}$ Compactifications (continued..)

S^1- compactifications of the bosonic string

$$\mathbb{R}^{1,25} \longrightarrow \mathbb{R}^{1,24} \times S^1_R$$

We have discussed this from two perspectives

① From the spacetime EFT:

Kaluza-Klein mechanism on the $\mathbb{R}^{1,25}$ EFT

We obtained a massless sector for a $\mathbb{R}^{1,24}$ EFT

$$G_{\mu\nu}(x) \longrightarrow G_{\mu\nu}(x^i) : \{ \underbrace{G_{ij}(x^i)}_{\text{25 dim graviton}},\ \underbrace{G_{i,25}(x^i)}_{\substack{\text{graviphoton} \\ e^{2\sigma} A}},\ \underbrace{G_{25,25}(x^i)}_{\substack{\text{radion} \\ e^{2\sigma}}} \}$$

$$B_{\mu\nu}(x) \longrightarrow B_{\mu\nu}(x^i) : \{ \underbrace{B_{ij}(x^i)}_{\text{25 dim KR field}},\ \underbrace{B_{i,25}(x^i)}_{\substack{\text{KR-photon} \\ \tilde{A}}} \}$$

2 $U(1)$ gauge symmetries

$$\Phi(x) \longrightarrow \Phi(x^i) \quad \text{25 dim dilaton}$$

$\boxed{\text{Plus}}$ a discrete infinite tower of massive states (KK-modes)

For example: the 26 dimensional dilaton gives rise to a discrete infinite tower of scalar fields (KK modes)

$$\phi_n \quad \text{with mass} \quad M_n^2 = \frac{n^2}{R^2} \qquad \forall n \in \mathbb{Z}$$

These are charged ($n \neq 0$) under the graviphoton:

charge $\quad \frac{n}{R} \quad$ (KK-momentum charge)

There are no modes charged under the $U(1)$ corresponding to the KR photon.

Of course this introduces a new scale

$$M_{KK} \sim \frac{1}{R}$$

In fact one can show that $\langle \sigma \rangle = R$
(see Blumenhagen + Lüst + Theisen)

We should not trust the EFT analysis for $M_{KK} \sim M_s$: however one can perform an exact analysis of the worldsheet CFT.

(2) The world sheet perspective

2 dim world sheet NLσM with target space with a nontrivial topology

$$X^i \longrightarrow X^i \qquad i = 0, \dots, 24$$

$$X^{25} \sim X^{25} + 2\pi R \qquad (X^{25} \text{ parametrises a circle } S^1_R)$$

states in the string Hilbert space are similar to those of $\mathbb{R}^{1,25}$ however we have now quantized KK-modes on S^1_R and quantized winding modes

The winding modes come from requiring

$$X^{25}(\tau, \sigma+\pi) = X^{25}(\tau, \sigma) + 2\pi R w \qquad w \in \mathbb{Z}$$

($X^{a}(\tau,\sigma)$ periodic up to $2\pi R w$)

The expansion of $X^i(\tau,\sigma)$ $\quad i=0,\dots,24$

is as for $\mathbb{R}^{1,25}$, but the expansion for X^{25} changes

$$X^{25}(\tau,\sigma) = x^{25} + 2\alpha' p^{25}\tau + 2Rw\sigma + \text{oscillator modes}$$

where the p^{25} momentum eigenvalue is quantized

$$p^{25} = \frac{m}{R} \qquad m \in \mathbb{Z}$$

Separating this into left & right movers:

$$X^{25}(\tau,\sigma) = X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-)$$

$$X_R^{25}(\sigma^-) = x_R^{25} + \left(\frac{\alpha' m}{R} - wR\right)\sigma^- + osc$$

$$X_L^{25}(\sigma^+) = x_L^{25} + \left(\frac{\alpha' m}{R} + wR\right)\sigma^+ + \widetilde{osc}$$

$$p_L^{25} = p^{25} + \frac{1}{\alpha'}Rw \qquad p_R^{25} = p^{25} - \frac{1}{\alpha'}Rw$$

The Virasoro operators are as before except

$$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(p^{25} + w\frac{R}{\alpha'}\right) \qquad \alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(p^{25} - w\frac{R}{\alpha'}\right)$$

The states are of the form:

$$\prod \alpha^{\mu}_{-n} \prod \tilde{\alpha}^{\nu}_{-m} \, | K ; m ; w \rangle$$

25-dim momentum evalue

$p^{25} = \frac{m}{R}$

$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R w$

and must satisfy the conditions: (apart from $\langle m | \phi \rangle = 0 \ \forall m > 0$)

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{R^2 w^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

mass shell condition

($M_{(25)}$ depends on R!)

contribution to the mass from momentum along the compact direction

contribution to the mass of the string winding around the circle $|w|$ times $\left(\frac{Rw}{\alpha'}\right)^2 = (2\pi R T w)^2$

$$N - \tilde{N} = m w$$

level-(mis)matching condition

Massless spectrum: for any R

state with $m = w = 0$

$M^2_{(25)} = -8$

- 25 dim graviton $\quad \gamma_{ij} \; \alpha^i_{-1} \tilde{\alpha}^j_{-1} \underbrace{|0, K\rangle \otimes |0,0\rangle}$
- 25 dim B-field $\quad B_{ij} \; \alpha^i_{-1} \tilde{\alpha}^j_{-1} |0, K\rangle \otimes |0,0\rangle$
- scalar from the trace part of γ: $\Phi_{(25)}$
- 2× 25 dim $U(1) \times U(1)$ gauge fields $\quad \xi \cdot (\alpha_{-1} \tilde{\alpha}^{25}_{-1} \pm \tilde{\alpha}_{-1} \alpha^{25}_{-1}) |0, K\rangle \otimes |0,0\rangle$

(graviphoton from the 26 dim metric + another photon from the 26 dim KR field)

- scalar ("radion") $\quad \alpha^{25}_{-1} \tilde{\alpha}^{25}_{-1} |0, K\rangle \otimes |0,0\rangle$

identified with the scalar σ

massless string spectrum $\longleftrightarrow$ massless spectrum from KK reduction of EFT

For certain values of (eg $\alpha' = \sqrt{R}$!) there are more.

Remark: we have introduced a new scale R

In fact, we have a one parameter family of compactifications with $R \in (0, \infty)$

R is called a modulus

(More general compactifications give rise to a moduli space)

However $(0, \infty]$ contains values of R which give rise to indistinguishable physical theories.

6.3 $\boxed{\text{T-duality}}$ (closed strings)

Returning to the mass formulas

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

$$N - \tilde{N} = mw$$

Limiting cases :

expect theory on $\mathbb{R}^{1,25}$

- $R \to \infty$: continuum of KK modes → sign of 25th dimension
  ($S^1_R \to \mathbb{R}$) $w = 0$

- $R \to 0$: continuum of winding modes ?
  $m = 0$

$\boxed{\text{Symmetry}}$ of the spectrum: observe that the formulas

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N+\tilde{N}-2) \,, \qquad N-\tilde{N} = mw$$

are invariant under: $m \leftrightarrow w$ & $R \leftrightarrow \frac{\alpha'}{R} = \hat{R}$

$\Rightarrow$ compactifications on S_R & $S_{\alpha'/R}$ have the $\boxed{\text{same}}$ spectrum.

[Note that $R = \sqrt{\alpha'}$ is a fixed point of this transformation: something special happens at this point.]

This is in fact an exact symmetry of the CFT

$$\boxed{\text{T-duality}}$$

as compactifications on S_R & $S_{\hat{R}}$ with $\hat{R} = \frac{\alpha'}{R}$ are indistinguishable as physical theories.

The interchange $m \leftrightarrow w$ means that

momentum excitations $\longleftrightarrow$ winding mode excitations

so continuum of KK modes $\longleftrightarrow$ continuum of winding modes
for $R \to \infty$ for $\hat{R} \to 0$

T-duality as an exact symmetry of the CFT

BUT, we have only shown that the spectrum is the same for two theories where

$$R \longleftrightarrow \hat{R} = \frac{\alpha'}{R}$$

and simultaneously

$$(m, w) \longleftrightarrow (w, m)$$

We need to consider the full CFT to prove this is an $\boxed{\text{exact}}$ symmetry of the CFT

Noting that $\begin{cases} \alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{m}{R} - \frac{Rw}{\alpha'}\right) \longrightarrow -\alpha_0^{25} \\ \tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} + \frac{Rw}{\alpha'}\right) \longrightarrow \tilde{\alpha}_0^{25} \end{cases}$ interchanges the graviphoton with the KR photon

we extend action of the transformation to the oscillator modes

$$X_R^{25}(\sigma^-) \longleftrightarrow -X_R^{25}(\sigma^-)$$
$$X_L^{25}(\sigma^+) \longleftrightarrow X_L^{25}(\sigma^+)$$

equivalently

$$X^{25}(\tau,\sigma) = X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-) \longleftrightarrow \hat{X}^{25}(\tau,\sigma) = X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-)$$

$$= x^{25} + 2\alpha' \frac{m}{R}\tau + 2Rw\sigma + \dots \qquad = \hat{x}^{25} + 2\alpha' \frac{m}{R}\sigma + 2Rw\tau + \dots$$

circle radius R
conj momentum $p^{25} = \frac{m}{R}$

circle radius $\frac{\alpha'}{R}$
conjugate momentum $\hat{p}^{25} = \frac{w\alpha'}{R}$

X & $\hat{X}$ have the same energy momentum tensor

$$T_{\pm\pm} = \partial_\pm X \cdot \partial_\pm X = \partial_\pm \hat{X} \, \partial_\pm \hat{X}$$

So one can recover L_m & $\tilde{L}_m$ as Fourier modes

$\Rightarrow$ CFTs of X & $\hat{X}$ are the $\boxed{\text{same}}$ with $\hat{R} = \frac{\alpha'}{R}$

As a consequence of this duality the moduli space of circle compactifications of the bosonic string is not $(0, \infty]$ but instead

$R \in (0, \sqrt{\alpha'}]$ or equivalently $R \in [\sqrt{\alpha'}, \infty)$

Fixed point of the duality transformation:

$$R \longleftrightarrow \hat{R} = \frac{\alpha'}{R} \quad \text{when} \quad \boxed{R = \sqrt{\alpha'}}$$

$R = \sqrt{\alpha'}$ is special $\rightsquigarrow$ more massless states and enhanced gauge symmetry

$R = \sqrt{\alpha'}$

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) = \frac{1}{R^2}\left(m^2 + w^2 + 2(N + \tilde{N} - 2)\right)$$

so $M^2_{25} = 0$ when $m^2 + w^2 + 2(N + \tilde{N}) = 4$

also $mw = N - \tilde{N}$

There are in fact 4 extra massless vectors which enhance the $U(1) \times U(1)$ symmetry to

$$SU(2) \times SU(2)$$

and 8 additional scalar fields in a $(\underline{3}, \underline{3})$ representation of $SU(2) \times SU(2)$

(BLT for details)

6.4 Open strings and T-duality

What happens to T-duality ?

Recall: open string boundary conditions compatible with Poincaré invariance in 26 dimensions

$$\frac{\partial}{\partial\sigma} X^\mu(\tau,\sigma) = 0 \quad \text{at} \quad \sigma = 0, \pi$$

Neumann boundary conditions

(ends of the string are free to move in spacetime)

Consider now compactifying on a circle

→ no winding modes !

while KK-momentum momentum modes still make sense compactify on a circle with X^{25} parametrising the circle of radius R.

Now consider what happens when interchanging

$$X^{25}_L \longleftrightarrow X^{25}_L \qquad X^{25}_R \longleftrightarrow -X^{25}_R$$

Should we expect a dual string for which there is a winding quantum number but no KK-momentum?

The proposed dual coordinate is

$$\hat{X}^{25}(\tau,\sigma) = X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-)$$

$$= \hat{x} + 2\alpha' p^{25}\sigma + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \hat{\alpha}_n e^{-in\tau} \sin(n\sigma)$$

$$= \hat{x} + 2\alpha' \frac{m}{R}\sigma + \text{osc} \qquad = \hat{x} + 2m\hat{R}\,\sigma + \text{osc}$$

- no term linear in τ ie the dual string has no momentum in the circle direction : translation invariance along S^1 is broken

- Moreover dual string wraps around the dual circle m times ?

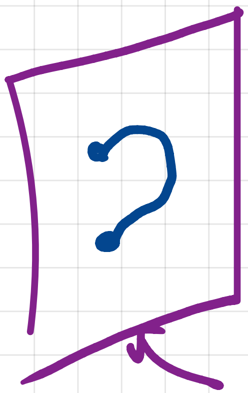
Boundary conditions of the dual string: at $\sigma = 0, \pi$

$$\left.\begin{aligned} \hat{X}^{25}(\tau,\sigma)\Big|_{\sigma=0} &= \hat{x} \\ \hat{X}^{25}(\tau,\sigma)\Big|_{\sigma=\pi} &= \hat{x} + 2\alpha' \frac{m}{R}\pi = \hat{x} + 2\pi m \hat{R} \end{aligned}\right\}$$

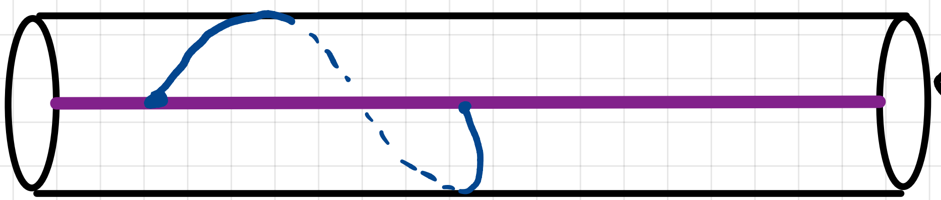
position of the end points of the dual string are fixed.

→ This is a Dirichlet boundary condition!

the dual open string is attached to a $(1+24)$ dimensional hyperplane, a $\boxed{\text{D24-brane}}$



25-dim plane



$S^1_{\hat{R}}$

Under a T-duality transformation:

open string with Neumann boundary condition on S'_R $\longleftrightarrow$ open string with Dirichlet boundary condition on $S'_{\hat{R}}$

[momentum $\frac{m}{R}$ along S'_R $\longleftrightarrow$ no momentum along $S'_{R'}$

no winding around S'_R $\longleftrightarrow$ winding around $S'_{R'}$]

The subspace where the string ends are attached to is called a D-brane

convention: a Dp-brane is a D-brane with p spatial dimensions

(so it is p+1 dimensional)

T-duality

$\longleftrightarrow$

open string with Neumann boundary conditions compactified on S^1_R

D25 space-filling brane

↳ open string ends are free to move on spacetime

$p^{25} = \frac{m}{R}$ quantized

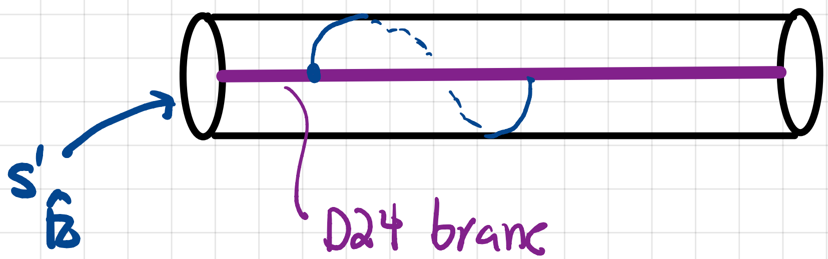
no winding

dual open string with Dirichlet boundary conditions compactified on $S^1_{\hat{R}}$, $\hat{R} = \alpha'/R$

endpoints of the string live on a D24 brane

no translational symmetry along $S^1_{\hat{R}}$

string can wind around $S^1_{\hat{R}}$



massless sector: (both sides)

25 dimensional U(1) gauge fields

Next : epilogue on D-branes