Problem Sheet 2

Section A

No work in this section will be marked. Guided solutions will be published. The material has to be considered as preliminary/bookwork.

QUESTION 1. **Mollification.**

(1) Give an example of a function (that will play later the role of kernel for mollification) with the following properties:
- $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\phi) = B_1(0)$;
- $\phi(x) \geq 0$ for all $x \in \mathbb{R}^n$;
- $\int_{B_1(0)} \phi(x)dx = 1$.

(2) Given ϕ as in point 1, for every function $u \in L^1_{loc}(\mathbb{R}^n)$ define

$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy.$$

Show that $u \star \phi \in C^\infty(\mathbb{R}^n)$.
Hint: observe that $\int_{\mathbb{R}^n} u(x-y)\phi(y)dy = \int_{\mathbb{R}^n} \phi(x-y)u(y)dy$.

(3) Given ϕ as in point 1, for every $\epsilon \in (0,1)$, let

$$\phi_\epsilon(x) := \epsilon^{-n}\phi(x/\epsilon).$$

Show that $\mathrm{supp}(\phi_\epsilon) = B_\epsilon(0)$ and that $\int_{B_\epsilon(0)} \phi_\epsilon(x)dx = 1$.

(4) If $u \in C(\mathbb{R}^n)$, show that $u \star \phi_\epsilon$ converges to u uniformly on compact subsets of $\mathbb{R}^n$.

Solution For more on mollification see the Lecture Notes of C4.3 "Functional Analytic methods for PDEs".

(1) Define $\phi(x) := 0$ for $|x| \geq 1$ and $\phi(x) := C\exp(\frac{1}{|x|^2-1})$ for $|x| < 1$, with $C > 0$ chosen such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. It is easily seen that such ϕ has all the desired properties.

(2) First of all we notice that, if $u \in L^1_{loc}(\mathbb{R}^n)$ then

$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy$$

is well defined for all $x \in \mathbb{R}^n$. By the change of variable $z = x - y$ we directly see that

$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy = \int_{\mathbb{R}^n} u(z)\phi(x-z)dz = \int_{\mathbb{R}^n} \phi(x-y)u(y)dy,$$

proving the hint. Now, since ϕ is C^1 with compact support, we can use the Differentiation Theorem (it is a corollary of Dominated Convergence Theorem) to infer that

$$\partial_{x_i}(u \star \phi)(x) = \partial_{x_i}\left(\int_{\mathbb{R}^n} \phi(x-y)u(y)dy\right) = \int_{\mathbb{R}^n} (\partial_{x_i}\phi)(x-y)u(y)dy.$$

This shows that $u \star \phi$ is C^1. By iterating the procedure, we obtain that $u \star \phi$ is C^∞.

(3) Follows directly from (1) by changing variables.

(4) If $u \in C(\mathbb{R}^n)$ then it is uniformly continuous on compact subsets. Fix a compact subset $K \Subset \mathbb{R}^n$. Using that $\phi_\epsilon \geq 0$, $\int \phi_\epsilon = 1$ and that $\mathrm{supp}(\phi_\epsilon) = B_\epsilon(0)$, for every $x \in K$ we have that

$$\begin{aligned} |u(x) - u \star \phi_\epsilon(x)| &= \left|\int_{\mathbb{R}^n} (u(x) - u(x-y))\,\phi_\epsilon(y)dy\right| \leq \int_{\mathbb{R}^n} |u(x) - u(x-y)|\,\phi_\epsilon(y)dy \\ &\leq \sup_{y \in \mathbb{R}^n, |y| \leq \epsilon} |u(x) - u(x-y)|\,. \end{aligned}$$

Denote with $K_1 := \{x \in \mathbb{R}^n : \text{there exists } y \in K \text{ such that } |x - y| \le 1\}$, (i.e. the set of points at distance at most 1 from K) and notice that K_1 is compact as well. From the previous estimate, we obtain

$$\sup_{x\in K} |u(x) - u \star \phi_\epsilon(x)| \le \sup_{x_1,x_2\in K_1, |x_1-x_2|\le\epsilon} |u(x_1) - u(x_2)| \to 0 \quad \text{as } \epsilon \to 0$$

by uniform continuity of u on the compact set K_1.

□

QUESTION 2. **An application of Brouwer's fixed point Theorem: zero's of continuous vector fields.**

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Assume that there exists $R > 0$ such that

$$g(x) \cdot x \ge 0, \qquad \text{for all } x \text{ with } |x| = R.$$

Show that there exists $x^* \in \overline{B_R(0)}$ such that $g(x^*) = 0$; in other words, show that the vector field g has a zero in $\overline{B_R(0)}$.

Hint: Argue by contradiction, consider the map $f(x) := -\frac{R}{|g(x)|} g(x)$ *and apply Brouwer's fixed point Theorem.*

Solution Assume that there exists no such x^*. Then we can define

$$f(x) = -R\frac{g(x)}{|g(x)|}.$$

f is continuous and $f\colon \overline{B_R(0)} \to \overline{B_R(0)}$. Brouwer's FPT implies that there exists $x_1 \in \overline{B_R(0)}$ such that $f(x_1) = x_1$. Then $|x_1| = |f(x_1)| = R$, and thus the assumption on g implies $g(x_1) \cdot x_1 \ge 0$.

On the other hand

$$g(x_1) \cdot x_1 = -f(x_1) \cdot x_1 \frac{|g(x_1)|}{R} = -\frac{|x_1|^2 |g(x_1)|}{R} < 0,$$

which is a contradiction.

□

Section B

Work done in this section will be marked.

QUESTION 3. **Leray-Schauder/Schaefer Theorem.**

- Prove the following result Let X be a Banach space and $T\colon X \to X$ be a compact map with the following property: there exists $R > 0$ such that the statement ($x = \tau T x$ with $\tau \in [0, 1)$) implies $\|x\|_X < R$. Then T has a fixed point x^* such that $\|x^*\|_X \le R$.
  Hint: Consider the operators
  $$T_n(x) := \begin{cases} Tx & \textit{if } \|Tx\|_X \le R + \frac{1}{n}, \\ \frac{R+1/n}{\|Tx\|_X} Tx & \textit{else} \end{cases}$$
  on a suitable domain and prove that they are compact.
- Let $T : X \to X$ a compact map such that there exists $R > 0$ such that $\|Tx - x\|_X^2 \ge \|Tx\|_X^2 - \|x\|_X^2$ when $\|x\|_X \ge R$. Show that T admits a fixed point.

QUESTION 4. **Integral operators on $L^2(\Omega)$ vs. $C(\bar{\Omega})$** As always, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain.

- Let $a : \bar{\Omega} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous map, and let
$$A(u)(x) = \int_\Omega a(x, y, u(y))dy.$$
show that $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined and compact. *Hint: use Arzela-Ascoli Theorem.*
- Let $k \in L^2(\Omega \times \Omega)$ and define
$$(Ku)(x) = \int_\Omega k(x, y)u(y)dy.$$
Show that $K : L^2(\Omega) \to L^2(\Omega)$ is well defined and compact. You can use for example that $C_0^\infty(\Omega \times \Omega)$ is dense in $L^2(\Omega \times \Omega)$, and therefore there is a sequence $k_m \in C_0^\infty(\Omega \times \Omega)$ such that $k_m \to k$ in $L^2(\Omega)$.
- Give an example of continuous a such that A (defined as above) is not well defined as an operator from $L^2(\Omega) \to L^2(\Omega)$.

QUESTION 5. **Continuous maps.** Let $g \in C(\mathbb{R} \times \mathbb{R}^n)$ be such that $g(z, p) \leq a + b|z|^\alpha + c|p|$, where a,b and c are non negative constants, and $2\alpha < 2^*$, where $2^* = 2n/(n-2)$ if $n \geq 3$, and $2* = \infty$ if $n = 1, 2$. Then the map $u \mapsto g(u, \nabla u)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and maps bounded subsets of $H_0^1(\Omega)$ to bounded subsets of $L^2(\Omega)$.

Hint: rewrite $g(u, \nabla u) = \tilde{g}(u, \frac{\nabla u}{|\nabla u|^\nu})$ *for a suitable function* $\tilde{f}$ *and a suitable exponent* $0 < \nu < 1$ *and apply Lemma 25 from the lecture notes.*

Section C

No work in this section will be marked. Guided solutions will be published. These problems are not more difficult than those in previous sections. They sit here simply because they are relevant but either slightly off or beyond the main interests of the course.

QUESTION 6. **Leray's eigenvalue problem**. Let $K : [a, b] \times [a, b] \to (0, \infty)$ be a continuous and positive function and consider the integral operator $T : C^0([a, b]) \to C^0([a, b])$ defined by
$$(Tu)(x) = \int_a^b K(x, t)u(t)\,dt.$$
Prove that T has at least one non-negative eigenvalue λ whose eigenvector is a continuous non-negative function u, i.e. there exist $\lambda \geq 0$ and a non-negative u so that
$$\int_a^b K(x, t)u(t)\,dt = \lambda u(x).$$
Hint: consider, on an appropriate closed convex set M*, the function*
$$F(u) = \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$
and apply one of the versions of Schauder's Fixed Point Theorem with the help of Arzéla-Ascoli Theorem. To find a suitable set M *think about what property all functions* $F(u)$ *have in common.*

Solution. Since $K : [a,b] \times [a,b] \to (0,\infty)$ is continuous, there exist $c_1, c_2 \in (0,\infty)$ such that

$$c_1 \le K(x,t) \le c_2\,, \quad \text{for all } (x,t) \in [a,b]^2.$$

We know from First year Analysis that if $u \in C^0([a,b])$, then the function $x \mapsto \int_a^b K(x,t)u(t)\,dt := Tu(x)$ is continuous on $[a,b]$ as well. Moreover, if $u \ge 0$ then we have

$$c_1 \int_a^b u(t)\,dt \le \int_a^b K(x,t)u(t)\,dt \le c_2 \int_a^b u(t)\,dt, \quad \text{for all } u \ge 0.$$

Consider now

$$F(u) := \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$

Observe that $\int_a^b (Fw)(t)\,dt = 1$ for every $w \ge 0$. Then, any fixed point of F will satisfy $u(x) = (Fu)(x)$ so in particular

$$\int_a^b u(t)\,dt = \int_a^b (Fu)(t)\,dt = 1.$$

Observe that

$$M := \left\{ u \in C^0([a,b]) \,:\, u \ge 0, \int_a^b u(t)\,dt = 1, \right\}$$

is convex, closed and non-empty. In order to apply Schauder Theorem version III, we need to prove that $F : M \to M$ is continuous and that $F(M)$ is compact.

Claim 1: $F : M \to M$ is continuous.
We know that $Tu(x) \ge c_1(b-a) > 0$. It easily follows that $Fu(x) \ge 0$ for all $x \in [a,b]$ as well. Moreover, we already observed that $\int_a^b (Fu)(t)\,dt = 1$, and thus F maps M to M.
Proof that $F : M \to M$ is continuous. since the map $u \mapsto K(u)$ is continuous on $C^0([a,b])$, so also the map $u \mapsto \int_a^b K(u)(t)\,dt$ is continuous. Moreover, this is bounded from below:

$$0 < c_1|b-a|^2 \le \int_a^b Tu(t)\,dt \le c_2|b-a|^2. \tag{1}$$

So claim 1 follows.

Claim 2: $F(M)$ is compact.
We first show that $F(M)$ is bounded. From (1) we obtain that

$$0 \le \frac{c_1}{c_2(b-a)} \le F(u)(t) \le \frac{c_2}{c_1(b-a)}$$

for all $u \in M$ and all $t \in [a,b]$. Thus $F(M)$ is bounded.
In order to show that $F(M)$ is pre-compact it is then enough to show that it is equi-continuous (so the pre-compactness will follow from Arzelá-Ascoli's Theorem).
Let $\delta > 0$, $t_{1,2}$ be such that $|t_1 - t_2| < \delta$ and let $u \in M$. Denote $\mu(u) := \int_a^b K(u)(t)\,dt$. Then

$$|F(u)(t_1) - F(u)(t_2)| \le \frac{1}{\mu(u)} \int |K(t_1,x) - K(t_2,x)|\,u(x)\,dx \le \frac{1}{\mu(u)} \sup_{x,|t_1-t_2|<\delta} |K(t_1,x) - K(t_2,x)|.$$

The conclusion follows by the uniform continuity of K on the compact set $[a,b]^2$ and by lower bound $\mu(u) \ge (b-a)^2 c_1 > 0$. $\square$