Analytic Topology

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Update 20.11.2017: fixed the definition of paracompactness by inserting 'open';

This document contains the (important) definitions, statements and proof sketches for results which require a new idea. The other proofs should be straightforward.

Concepts and results which should be known from a previous course are typeset in a smaller size.

Throughout this document assume that X, Y, Z as well as $X_i, Y_i, Z_i, i \in I$ are topological spaces and I is some index set unless otherwise indicated.

1 Topological spaces, Bases, Subbases, Initial Topology, Products

1.1 Definitions

A topology on a set X is a collection τ of subsets of X containing the empty set, X that is closed under taking finite intersections and arbitrary unions.

A topological space is a pair of (X, τ) such that τ is a topology on X.

In a topological space (X, τ), elements of X are called points, elements of τ are called open sets, complements of elements of τ are called closed sets and subsets of X that are closed and open are called clopen.

For a subset A of a topological space X, the closure of A, $\overline{A}$, is the smallest closed set containing A and the interior of A, $\operatorname{int}(A)$, is the largest open set contained in A.

A function $f: X \to Y$ is continuous if and only if preimages of Y-open sets under f are X-open.

If $A \subseteq X$, the subspace topology on A is $\{U \cap A : U \text{ open } \subseteq X\}$.

A basis for a topology τ on X is a collection $\mathcal{B} \subseteq \tau$ such that every open set is a union of a subcollection $\mathcal{B}'$ of $\mathcal{B}$. If a basis has been fixed, its elements are called basic open sets.

X is metrizable if and only if there is a metric d on X such that $\left\{B^d_\epsilon(x) \colon x \in X, \epsilon > 0\right\}$ is a basis for X.

A space is second countable if and only if it has a countable basis.

A subbasis for a topology τ on X is a collection $\mathcal{S} \subseteq \tau$ such that the set of finite intersections of elements of $\mathcal{S}$ is a basis for τ. If a subbasis has been fixed, its elements are called subbasic open sets.

Given a set and a collection $\mathcal{F} = \{f_i \colon X \to Y_i \colon i \in I\}$the initial topology with respect to $\mathcal{F}$ is the smallest (wrt $\subseteq$) topology on X such that each $f_i \in \mathcal{F}$ is continuous.

The Tychonoff product $\prod_{i \in I} X_i$ of topological spaces $X_i, i \in I$ is the topological space consisting of the Cartesian (set) product equipped with the initial topology with respect to the projections.

1.2 Results

Lemma 1.1 (Recall)**.** *1. If $A \subseteq X$, then $\overline{A}$ exists and equals*

$$\bigcap \{C \colon A \subseteq C \text{ closed } \subseteq X\} = \{x \in X \colon \forall \text{ open } U \ni x \; U \cap A \neq \emptyset\} .$$

2. The closure operator $A \mapsto \overline{A}$ satisfies $\overline{\emptyset} = \emptyset$, $\overline{\overline{A}} = \overline{A}$, $A \subseteq \overline{A}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{\bigcap_i A_i} \subseteq \bigcap_i \overline{A_i}$. Dual results hold for the interior operator.

3. A function $f \colon X \to Y$ is continuous if and only if for every $A \subseteq X$, $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.

4. If $B \subseteq A \subseteq X$ then $\overline{B}^A = \overline{B}^X \cap A$.

Theorem 1.2. *1. The set of topologies on a fixed set X is a complete lattice with respect to $\subseteq$, i.e. a partial order with arbitrary infima and suprema. The infimum of a collection $\tau_i, i \in I$ of topologies on X is $\bigcap_i \tau_i$. The greatest element of the complete lattice is the discrete topology $\mathcal{P}(X)$, the smallest element is the indiscrete topology $\{\emptyset, X\}$.*

2. A collection $\mathcal{B}$ of subsets of X is the basis for a (necessarily unique) topology $\tau = \{\bigcup \mathcal{B}' \colon \mathcal{B}' \subseteq \mathcal{B}\}$ on X if and only if $\bigcup \mathcal{B} = X$ and for every $B_1, B_2 \in \mathcal{B}$ there is $\mathcal{B}' \subseteq \mathcal{B}$ such that $B_1 \cap B_2 = \bigcup \mathcal{B}'$. Moreover, τ is the smallest topology on X containing $\mathcal{B}$.

3. Every collection $\mathcal{S}$ is a subbasis for a (necessarily unique) topology τ on X with basis $\{\bigcap \mathcal{F} \colon \mathcal{F} \text{ finite } \subseteq \mathcal{S}\}$. Moreover, τ is the smallest topology on X containing $\mathcal{S}$.

4. *If Y is a topological space with a fixed subbasis, a function $f\colon X \to Y$ is continuous if and only if preimages of subbasic open sets under f are open.*

5. *If X is a set, $Y_i, i \in I$ are topological spaces and $f_i\colon X \to Y_i$ are functions, the initial topology with respect to the f_i exists and has subbasis $\left\{ {f_i}^{-1}(U) : i \in I, U \text{ open } \subseteq Y_i \right\}$. It is the unique topology on X such that for every topological space Z and every function $f\colon Z \to X$, f is continuous if and only if each $f_i \circ f$ is continuous.*

6. *The product topology on $\prod_i X_i$ has basis*

$$\left\{ \prod_i U_i\colon U_i \text{ open } \subseteq X \text{ and } X_i = U_i \text{ except for finitely many } i \right\}.$$

7. ***Embedding Lemma:*** *If $f_i\colon X \to X_i$ are continuous maps such that for distinct $x, y \in X$ there is $i \in I$ with $f_i(x) \neq f_i(y)$ and such that $\left\{ {f_i}^{-1}(U) : i \in I, U \text{ open } \subseteq X_i \right\}$ is a basis for X then the diagonal $\Delta = \Delta_i f_i\colon X \to \prod_i X_i; x \mapsto (f_i(x))_i$ is a homeomorphic embedding.*

8. *Countable products of metrizable spaces are metrizable.*

1.3 Proofs

Most of the proofs are straightforward set arithmetic.

5. For uniqueness, suppose that τ_1, τ_2 are two topologies on X satisfying the condition. Then $\mathrm{id}_{1,1}\colon (X, \tau_1) \to (X, \tau_1)$ is continuous, so each $f_i\colon (X, \tau_1) \to Y_i = f_i \circ \mathrm{id}_{1,1}$ is continuous. Thus every $f_i \circ \mathrm{id}_{1,2}$ is continuous and hence $\mathrm{id}_{1,2}$ is continuous. By symmetry $\mathrm{id}_{2,1}$ is continuous and hence $\tau_1 = \tau_2$.

7; **Embedding Lemma:** The only non-trivial bit is to check that Δ is open onto its image. For this note that unions and images commute and hence it is sufficient to consider basic open sets of the form ${f_i}^{-1}(U)$. But $\Delta\left({f_i}^{-1}(U)\right) = {\pi_i}^{-1}(U) \cap \Delta(X)$.

8. Exercise Sheet.

2 Separation Properties

2.1 Definitions

X is T_0 if and only if for every distinct $x, y \in X$ there is open U that contains exactly one of x and y.

X is T_1 if and only if for every distinct $x, y \in X$ there is open U such that $x \in U \not\ni y$.

X is T_2 (Hausdorff) if and only if for every distinct $x, y \in X$ there are disjoint open $U \ni x, V \ni y$ (x and y are separated by open sets).

X is T_3 (regular) if and only if X is T_1 and for every $x \in X$ and every closed $C \not\ni x$ there are disjoint open $U \ni x, V \supseteq C$.

X is $T_{3.5}$ (Tychonoff) if and only if X is T_1 and for every $x \in X$ and every closed $C \not\ni x$ there is a continuous $f \colon X \to [0,1]$ such that $f(x) = 0$ and $f(C) \subseteq \{1\}$.

X is T_4 (normal) if and only if X is T_1 and for every disjoint closed C, D there are disjoint open $U \supseteq C, V \supseteq D$.

X is functionally normal if and only if X is T_1 and for every disjoint closed C, D there is a continuous function $f \colon X \to [0,1]$ such that $f(U) \subseteq \{0\}, f(V) \subseteq \{1\}$.

X is T_5 (hereditarily normal) if and only if every subspace of X is normal.

X is T_6 (perfectly normal) if and only if X is T_1 and for every closed subspace C of X there is a continuous $f \colon X \to [0,1]$ such that $C = f^{-1}(\{0\})$.

2.2 Results

Theorem 2.1. *1. X is T_1 if and only if every singleton is closed.*

2. If a basis for X has been fixed then for $i \leq 2$, replacing 'open' in the definition of T_i by 'basic open' yields an equivalent property.

3. functionally normal $\implies T_{3.5} \implies T_3 \implies T_2 \implies T_1 \implies T_0$. (None of these reverse in general.)

4. X is Tychonoff if and only if X is (homeomorphic to) a subspace of a power of $[0,1]$.

5. For $i \leq 3.5$, products and subspaces of T_i-spaces are T_i.

*6. **Urysohn's Lemma:** Functionally Normal $\iff T_4$.*

7. *Subspaces of normal spaces need not be normal. Products (even squares) of normal spaces need not be normal.*

8. ***Urysohn's Metrization Theorem I:*** *If X is normal and second countable then X is metrizable.*

9. *Metric spaces are perfectly normal.*

10. $T_6 \implies T_5 \implies T_4$ *(**Not examinable as bookwork.**)*

11. *A normal space is perfectly normal if and only if every closed subset is a countable intersection of open subsets (a G_δ).(**Not examinable as bookwork.**)*

2.3 Proofs

General important ideas are:

- A, B disjoint is equivalent to $A \subseteq X \setminus B$ (and of course B is open if and only if $X \setminus B$ is closed).
- If $f \colon X \to [0,1]$ is continuous, then $f^{-1}([0,1/3)), f^{-1}((2/3,1])$ are disjoint open and $f^{-1}(0) = \bigcap_n f^{-1}([0, 2^{-n}))$.

4. Apply the Embedding Lemma.

5. For productivity of Tychonoffness, let $x \in U_1 \times \cdots \times U_n \times \prod X_i$. For each $k = 1, \ldots, n$, find a continuous f_k which is 1 at $\pi_k(x)$ and 0 outside U_i and take the product of the $f_k \circ \pi_k$.

6. **Urysohn's Lemma:** Backwards direction: Well order the countable set $\mathbb{Q} \cap (0,1)$, set $\overline{U_0} = C$, $U_1 = X \setminus D$ and inductively construct open U_r such that $r < s \implies \overline{U_r} \subseteq U_s$. Now define $f \colon X \to [0,1]$ by $f(x) = \sup\{r \colon x \in U_r\}$, note that $f(x) = \sup\left\{r \colon x \in \overline{U_r}\right\}$ and hence that $f(x) > \alpha$ if and only if there is $r \in \mathbb{Q} \cap (\alpha, 1]$ such that $x \in U_r$ and $f(x) < \alpha$ if and only if there is $r \in \mathbb{Q} \cap [0, \alpha)$ such that $x \in X \setminus \overline{U_r}$. Thus $f^{-1}((\alpha, 1]) = \bigcup_{r > \alpha} U_r$ and $f^{-1}([0, \alpha)) = \bigcup_{r < \alpha} X \setminus \overline{U_r}$ gives continuity.

7 We let $Y_f = \aleph_1 \cup \{\star\}$ with topology $\mathcal{P}(\aleph_1) \cup \{Y \setminus C \colon C \text{ finite } \subseteq \aleph_1\}$. It is easy to check that this is normal (it is compact Hausdorff) and we let $X = \{0\} \cup \{2^{-n} \colon n \in \mathbb{N}\}$ (with its usual topology). Then $Y_f \times X$ is

compact Hausdorff so normal. Consider the subspace $Y_f \times X \setminus \{(\star, 0)\}$. $C = \aleph_1 \times \{0\}$ and $D = \{\star\} \times \{2^{-n} : n \in \mathbb{N}\}$ are closed disjoint. If $U \supseteq D$, then for each $n \in \mathbb{N}$ there is a countable $C_n \subseteq \aleph_1$ such that $(\aleph_1 \setminus C_n) \times \{2^{-n}\} \subseteq U$. Pick $\alpha \in \aleph_1 \setminus \bigcup_n C_n$ (this is non-empty at $\bigcup_n C_n$ is countable) and note that $\{\alpha\} \times \{2^{-n} : n \in \mathbb{N}\} \subseteq U$. Thus $(\alpha, 0) \in \overline{U} \cap C$.

8. **Urysohn's Metrization Theorem I:** Let $\mathcal{B}$ be a countable basis and for each $(B, B') \in \mathcal{B}^2$ such that $\overline{B} \subseteq B'$ find a continuous $f : X \to [0, 1]$ such that $\overline{B} \subseteq f^{-1}(0), X \setminus B' \subseteq f^{-1}(1)$ and apply the Embedding Lemma to these (countably many) f.

9. If $C \subseteq X$ is closed then $d_C(x) = \inf \{d(x, c) : c \in C\}$ is as required.

3 Filters

3.1 Definitions

Suppose X is a set.

A filter $\mathcal{F}$ on X is a non-empty collection of subsets of X that does not contain $\emptyset$ and is closed under supersets and finite intersections.

A filter basis $\mathcal{B}$ for a filter $\mathcal{F}$ on X is a subcollection of $\mathcal{F}$ such that for every $F \in \mathcal{F}$ there is $\mathcal{B} \in \mathcal{B}$ with $B \subseteq F$.

Two collections $\mathcal{A}, \mathcal{B}$ of subsets of X mesh, written $\mathcal{A} \# \mathcal{B}$ if and only if for every $A \in \mathcal{A}, B \in \mathcal{B}$ we have $A \cap B \neq \emptyset$. We also write $\mathcal{A} \# \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

An ultrafilter on X is a filter on X that is maximal wrt $\subseteq$.

For $x \in X$, the principal filter at x is $\mathcal{P}_x = \{A \subseteq X : x \in A\}$.

If $f : X \to Y$ is a function and $\mathcal{F}$ is a filter on X then $f(F) := \{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\}$.

Now assume that X is a topological space.

For $x \in X$, the neighbourhood filter at x is $\mathcal{N}_x = \{N \subseteq X : \exists \text{ open } U \ x \in U \subseteq N\}$.

If $\mathcal{F}$ is a filter on X, $\lim \mathcal{F} = \{x \in X : \mathcal{N}_x \subseteq \mathcal{F}\}$ and $\mathcal{F} \to x$ if and only if $x \in \lim \mathcal{F}$.

If $\mathcal{P}$ is a property of topological spaces then X is locally $\mathcal{P}$ if and only if every neighbourhood filter has a filter basis of sets that are $\mathcal{P}$ (with respect to the subspace topology).

3.2 Results I

Lemma 3.1. *Suppose X is a set.*

1. *A non-empty collection $\mathcal{B}$ of non-empty subsets of X is a filter basis for a (necessarily unique) filter $\mathcal{F}$ on X if and only if for every $B_1, B_2 \in \mathcal{B}$ there is $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.*

2. *If $\mathcal{C}$ is a family of subsets of X with the f.i.p. then $\{\bigcap \mathcal{F} : \mathcal{F} \text{ finite } \subseteq \mathcal{C}\}$ is a filter basis for the smallest filter containing $\mathcal{C}$.*

3. *A filter $\mathcal{U}$ on X is an ultrafilter if and only if for every $A \subseteq X$ exactly one of A and $X \setminus A$ belongs to $\mathcal{U}$ if and only if whenever $A \cup B \in \mathcal{U}$ at least one of A or B belongs to $\mathcal{U}$.*

4. ***Ultrafilter Extension Lemma:*** *Every filter can be extended to an ultrafilter.*

5. *If $f : X \to Y$ is a function and $\mathcal{F}$ a filter on X then $f(F)$ is a filter on Y with filter basis $\{f(F) : F \in \mathcal{F}\}$. Moreover, if $\mathcal{F}$ is an ultrafilter then so if $f(F)$.*

3.3 Proofs I

Most of this is easy (once comfortable with the notation) set arithmetic.

General important ideas are:

- A family of subsets of X with the f.i.p. can be extended to a filter.
- If two families $\mathcal{A}, \mathcal{B}$ mesh, their mesh $\mathcal{A}\#\mathcal{B}$ can be extended to a filter.

Specific Lemmas:

3. Suppose $\mathcal{U}$ is an ultrafilter. If $A \notin \mathcal{U}$, then $X \setminus A \# \mathcal{U}$, so by maximality $X \setminus A \in \mathcal{U}$. The converse is obvious. For the last if and only if: for the forward direction assume $A \notin \mathcal{U}$. Then $X \setminus A, A \cup B \in \mathcal{U}$, so $B \supseteq (X \setminus A) \cap (A \cup B) \in \mathcal{U}$. For the backwards direction note that $A \cup (X \setminus A) = X \in \mathcal{U}$.

4. **Ultrafilter Extension Lemma: Proof not examinable!** Note that the union of an increasing sequence of filters is a filter and apply Zorn's Lemma.

3.4 Results II

Theorem 3.2. *Suppose X is a topological space.*

1. *Suppose $A \subseteq X$ and $x \in X$. $x \in \overline{A}$ if and only if there is a filter $\mathcal{F} \ni A$ such that $\mathcal{F} \to x$ if and only if there is an ultrafilter $\mathcal{U} \ni A$ such that $\mathcal{U} \to x$.*

2. *$f\colon X \to Y$ is continuous if and only if for every $x \in X$ and filter $\mathcal{F} \to x$, $f(F) \to f(x)$ if and only if for every $x \in X$ and ultrafilter $\mathcal{U} \to x$, $f(U) \to f(x)$.*

3. *X is Hausdorff if and only if every filter converges to at most one point if and only if every ultrafilter converges to at most one point.*

4 Compactness, Compactifications, Local Compactness, Čech-completeness

4.1 Definitions

A topological space is compact if and only if every open cover has a finite subcover.

A topological space is Lindelöf if and only if every open cover has a countable subcover.

A Hausdorff compactification of a topological space X is a pair (h, Y) where Y is compact Hausdorff and $h\colon X \to Y$ is a dense homeomorphic embedding.

For a topological space X and two Hausdorff compactification (h_1, Y_1), (h_2, Y_2), we define $(h_2, Y_2) \leq (h_1, Y_1)$ if and only if there is a continuous $g\colon Y_1 \to Y_2$ such that $g \circ h_1 = h_2$. We define $(h_1, Y_1) \sim (h_2, Y_2)$ if and only if there is a homeomorphism $g\colon Y_1 \to Y_2$ such that $g \circ h_1 = h_2$.

For a topological space X and a Hausdorff compactification (h, Y) we say that (h, Y) satisfies the Stone-Čech-property with respect to continuous maps into compact Hausdorff spaces if and only if for every compact Hausdorff space Z and every continuous $f\colon X \to Z$ there is a continuous $F\colon Y \to Z$ such that $f = F \circ h$.

The Stone-Čech compactifiation of a topological space is (the unique, if it exists) Hausdorff compactification $(\beta, \beta X)$ of X satisfying the Stone-Čech property wrt continuous maps into compact Hausdorff spaces.

Recall that a space X is locally compact if and only if for every $x \in U$ open $\subseteq X$ there is compact K and open V with $x \in V \subseteq K \subseteq U$.

The Alexandroff one-point compactification of a topological space X is (the unique, if it exists) Hausdorff compactfication $(\omega, \omega X)$ of X such that $\omega X \setminus \omega(X)$ is a singleton.

A Tychonoff topological space is Čech-complete if and only if for every Hausdorff compactification (h, Y) of X, $Y \setminus h(X)$ is a countable union of closed subsets of Y (i.e. an F_σ).

4.2 Results

The key ideas are:

- Compactness properties are inherited by closed subsets.
- Diagonals!
- if $f\colon Y \to Z$ is continuous and X dense in Y then $f(Y) \subseteq \overline{f(X)}$.
- Compactness is preserved by images, closedness by pre-images.

Lemma 4.1 (Recall)**.**
1. *A topological space is compact if and only if every family of closed sets with the finite intersection property has non-empty intersection.*
2. *Every closed subset of a compact topological space is compact.*
3. *Every compact subset of a Hausdorff space is closed.*
4. *Every compact Hausdorff space is regular. Every compact regular space is normal.*
5. *If X is compact, Y is a topological space and $f\colon X \to Y$ is continuous then $f(X)$ is compact.*

Theorem 4.2.
1. *Every Lindelöf regular space is normal.*
2. *Every second countable space is Lindelöf.*
3. *Every Lindelöf metric space is second countable.*
4. ***Urysohn's Metrization Theorem II:** A compact Hausdorff space is metrizable if and only if it is second countable.*
5. *X is compact if and only if every ultrafilter on X converges (to some point).*

6. ***Tychonoff's Theorem:*** *Products of compact spaces are compact.*

7. *X has a Hausdorff compactification if and only if it is Tychonoff.*

8. *$\sim$ is an equivalence relation on the Hausdorff compactifications of X. If (h_1, Y_1), (h_2, Y_2) are Hausdorff compactifications of X such that $(h_1, Y_1) \leq (h_2, Y_2) \leq (h_1, Y_1)$ then $(h_1, Y_1) \sim (h_2, Y_2)$ and thus $\leq$ induces a partial order on the equivalence classes of Hausdorff compactifications under $\sim$.*

9. *If (h_1, Y_1), (h_2, Y_2) are Hausdorff compactifications of X such that $(h_1, Y_1) \leq (h_2, Y_2)$ as witnessed by $g\colon Y_2 \to Y_1$ then $g\left(Y_2 \setminus h_2(X)\right) = Y_1 \setminus h_1(X)$.*

10. *If X is Tychonoff, then the partial order of (equivalence classes of) Hausdorff compactifications has suprema. Moreover each equivalence class has a representative with cardinality $2^{2^{|X|}}$.*

11. *If X is Tychonoff, then X has a Stone-Čech compactification which is unique (up to equivalence) and is the greatest compactification of X.*

12. *A Hausdorff compactification of X satisfies the Stone-Čech property with respect to continuous maps into compact Hausdorff spaces if and only if it satisfies the Stone-Čech property with respect to continuous maps into $[0, 1]$.*

13. *Open subsets of locally compact spaces are locally compact.*

14. *Compact Hausdorff spaces are locally compact.*

15. *If X is non-compact, locally compact, Hausdorff and $\infty \notin X$ then $\omega X = X \cup \{\infty\}$ with topology $\{U\colon U \text{ open } \subseteq X\} \cup \{\omega X \setminus K\colon K \text{ compact } \subseteq X\}$ and embedding $\omega\colon X \to \omega X; x \mapsto x$ is the unique one-point compactification of X.*

16. *If X is Tychonoff, the following are equivalent:*

    - *X is locally compact.*
    - *X has a smallest Hausdorff compactification.*
    - *X has a one-point compactification.*

- *$\beta X \setminus \beta(X)$ is closed.*
- *For every Hausdorff compactification (h, Y) of X, $Y \setminus h(X)$ is closed.*
- *For some Hausdorff compactification (h, Y) of X, $Y \setminus h(X)$ is closed.*

17. *If X is Tychonoff, the following are equivalent:*
    - *$\beta X \setminus \beta(X)$ is a F_σ.*
    - *X is Čech-complete.*
    - *For some Hausdorff compactification (h, Y) of X, $Y \setminus h(X)$ is a F_σ.*

4.3 Proofs

1. Exercise Sheet.

5. Suppose X is compact and $\mathcal{U}$ is an ultrafilter not converging to any $x \in X$. For each $x \in X$, choose open $U_x \ni x$ such that $U_x \notin \mathcal{U}$. Then $\{U_x : x \in X\}$ is an open cover with finite subcover $U_{x_1}, \ldots, U_{x_n}$. Thus one of $U_{x_i} \in \mathcal{U}$ (a contradiction).

   Now assume that X is not compact: let $\mathcal{C}$ be a family of closed subsets with the f.i.p. but empty intersection. Extend $\mathcal{C}$ to a filter and then to an ultrafilter $\mathcal{U}$. If $x \in X$ then $x \notin$ some C_x, so $x \in X \setminus C_x \in \mathcal{N}_x$ and hence $\mathcal{N}_x \not\subseteq \mathcal{U}$, i.e. $\mathcal{U} \not\to x$.

6. **Tychonoff's Theorem:** Let $\mathcal{U}$ be an ultrafilter on $\prod_i X_i$. For each $i \in I$, $\pi_i(\mathcal{U})$ is an ultrafilter on X_i, so converges to some x_i. Now check that $\mathcal{U} \to (x_i)_i$.

8. If $(h_1, Y_1) \leq (h_2, Y_2) \leq (h_1, Y_1)$ is witnessed by $g\colon Y_2 \to Y_1$ and $h\colon Y_1 \to Y_2$ respectively, note that $g \circ h$ and $h \circ g$ are the identity on $h_1(X)$ and $h_2(X)$ respectively. But $h_i(X)$ is dense in the Hausdorff Y_i, so $g \circ h$ and $h \circ g$ are the identity on Y_1 and Y_2 respectively. Hence g is a homeomorphism as required.

9. Wlog h_2 is the identity. Suppose $y \in Y_2 \setminus X$ and $x \in X$ with $g(y) = h_1(x)$. Let $\mathcal{F}$ be a filter on Y_2 containing X and converging to $y \in \overline{X}$.

As Y_2 is Hausdorff $\mathcal{F} \not\to x$. Then $\mathcal{F}_X = \{F \cap X : F \in \mathcal{F}\}$ is a filter on X and $\mathcal{F}_X \not\to x$. As h_1 is a homeomorphism $X \to h_1(X)$, $h_1(\mathcal{F}_X) \not\to h_1(x) = g(y)$. But $h_1(\mathcal{F}_X) = g(\mathcal{F})$ and $g(\mathcal{F}) \to g(y)$ by continuity.

10. If (h_i, Y_i) are compactifications, then check that $(\Delta_i h_i, \overline{\Delta_i h_i(X)}^{\prod_i Y_i})$ is an upper bound. If (g, Z) is another upper bound witnessed by the $g_i \colon Z \to Y_i$, then $\Delta_i g_i \colon Z \to \prod_i Y_i$ is continuous and into $\overline{\Delta_i h_i(X)}$ since $\Delta_i g_i(Z) = \Delta_i g_i\left(\overline{g(X)}^Z\right) \subseteq \overline{\Delta_i g_i(g(X))} = \overline{\Delta_i h_i(X)}$.

    For the 'moreover' claim: suppose X is dense in Y and let $f \colon Y \to \mathcal{P}(\mathcal{P}(X)); y \mapsto \left\{A \subseteq X : y \in \overline{A}^Y\right\}$. As Y is Hausdorff, for $y \neq y'$ there is open $U \ni y$ with $y' \notin \overline{U}^Y = \overline{U \cap X}^Y$. Thus f is an injection.

11. First uniqueness up to $\sim$: suppose (h_1, Y_1), (h_2, Y_2) are Hausdorff compactifications of X satisfying the Stone-Čech property. Then $h_2 \colon X \to Y_2$ is a continuous map into a compact Hausdorff space, so there is $H_2 \colon Y_1 \to Y_2$ such that $H_2 \circ h_1 = h_2$, i.e. $(h_2, Y_2) \leq (h_1, Y_1)$. By symmetry $(h_1, Y_1) \leq (h_2, Y_2)$ and hence $(h_1, Y_1) \sim (h_2, Y_2)$.

    Now we show existence, by showing that the greatest Hausdorff compactification of X satisfies the Stone-Čech property: for each equivalence class, choose a representative, and let $(\beta, \beta X)$ be the supremum over these representatives (there are only set many). If $f \colon X \to Z$ is continuous into a compact Hausdorff Z, then $f\delta\beta \colon X \to \overline{f\delta\beta(X)}^{Z \times \beta X} = Y$ is an embedding by the Embedding Lemma and hence determines a Hausdorff compactification of X. As $(\beta, \beta X)$ is the greatest Hausdorff compactification of X, there is a continuous $g \colon \beta X \to Y$ with $g \circ \beta = f\delta\beta$. Then $F = \pi_Z \circ g$ is as required.

    Note that the existence proof has a special case ($f = h \colon X \to Y$) that shows that any compactification with the Stone-Čech property must be the greatest compactification of X (up to equivalence).

12. It is enough to check that the Stone-Čech property for continuous $[0,1]$-valued maps implies the Stone-Čech property for continuous maps into compact Hausdorff spaces. To that end, note that every compact Hausdorff space is normal, so Tychonoff, so homeomorphic to a closed subspace C of $[0,1]^I$ (for some I). So assume (h, Y) satisfies the Stone-Čech property for continuous $[0,1]$-valued maps and wlog

$X \subseteq Y, h = \mathrm{id}_X$. If $f\colon X \to C$ is continuous, then each $f_i = \pi_i \circ f$ extends to some $F_i\colon Y \to [0,1]$ and thus $\Delta_i f_i = f\colon X \to [0,1]^I$ extends to $\Delta = \Delta_i F_i\colon Y \to [0,1]^I$. But $\Delta(Y) = \Delta\left(\overline{X}\right) \subseteq \overline{\Delta(X)} \subseteq \overline{C} = C$. Hence Δ is as required.

16. Statements 1 and 3 are equivalent to 6. To see 4,5,6 are equivalent, we use that remainders map (on)to remainders: If $X \subseteq Y$ and Y compact Hausdorff with $Y \setminus X$ closed, then note that $\beta X \setminus \beta(X) = g^{-1}(Y \setminus X)$ where g witnesses $(\mathrm{id}, Y) \leq (\beta, \beta X)$ so that $\beta X \setminus \beta(X)$ is closed. If $\beta X \setminus \beta(X)$ is closed, it is compact and hence $Y \setminus X = g(\beta X \setminus \beta(X))$ is compact so closed where $X \subseteq Y$, Y is compact Hausdorff and g witnesses $(\mathrm{id}_X, Y) \leq (\beta, \beta X)$ giving 4 implies 5. Finally note that X is Tychonoff, so 5 implies 6 as X has a compactification.

    So assume one (hence all) of 1,3,4,5,6, let $(\omega, \omega X)$ be the one-point compactification of X and (h, Y) some Hausdorff compactification: we claim that $g\colon Y \to \omega X$ given by $g(h(x)) = \omega(x)$ and $g(y) = \star$ for $y \in Y \setminus h(X)$ is continuous: if C is closed in ωX then either $C \subseteq X$ and hence $g^{-1}(C) = h(C)$ is closed in $h(X)$ which is closed in Y or $C \ni \star$ and hence $g^{-1}(C) = (Y \setminus h(X)) \cup h(C \cap X)$ is a union of two closed sets, so closed.

    Finally assume 2 and that there is a smallest compactification (h, Y) that is not the one-point compactification: let $y_1, y_2 \in Y \setminus h(X)$ be distinct. Then $Y \setminus \{y_1, y_2\}$ is locally compact, Tychonoff and hence has a one-point compactification $Z = (Y \setminus \{y_1, y_2\}) \cup \{\star\}$. Since Y is a two-point compactification of $Y \setminus \{y_1, y_2\}$, there is a continuous $g\colon Y \to Z$ (which is the identity except that $g(y_1) = g(y_2) = \star$). But (h, Z) is also a Hausdorff compactificationof X so that there is and g witnesses that $(h, Z) \leq (h, Y)$. Thus the g above must be a homeomorphism, a contradiction to it not being injective.

17. Just like the equivalence of 4,5,6 in 16., noting that unions and images as well as pre-images commute.

5 Paracompactness, Bing's Metrization Theorem

5.1 Definitions

A family $\mathcal{A}$ of subsets of X is locally finite (resp. discrete) if and only if for every $x \in X$ there is open $U \ni x$ such that $\{A \in \mathcal{A} \colon U \cap A \neq \emptyset\}$ is finite (resp. empty or a singleton).

A family $\mathcal{A}$ of subsets of X is closure preserving if and only if for every $\mathcal{A}' \subseteq \mathcal{A}$, $\overline{\bigcup_{A \in \mathcal{A}'} A} = \bigcup_{A \in \mathcal{A}'} \overline{A}$.

A family $\mathcal{A}$ is a refinement of a family $\mathcal{B}$ (of subsets of X) if and only if for every $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $A \subseteq B$.

A topological space is paracompact if and only if every open cover $\mathcal{U}$ of X has a locally finite open refinement covering X.

The hedgehog of spininess κ is $H_\kappa = \{0\} \cup ((0,1] \times \kappa)$ with metric d given by $d(0,(t,i)) = t$, $d((t,i),(s,i)) = |t - s|$, $d((t,i),(s,j)) = t + s$ for $i \neq j$.

5.2 Results

Theorem 5.1. *1. Locally finite families are closure preserving.*

2. *A paracompact regular space is normal.*

3. *For a regular space X the following are equivalent:*

   - *X is paracompact.*
   - *Every open cover $\mathcal{U}$ has a sequence $\mathcal{V}_n$ of locally finite, open refinements such that $\bigcup_n \mathcal{V}_n$ covers X (i.e. every open cover has a σ-locally finite open refinement covering X).*
   - *Every open cover $\mathcal{U}$ has a locally finite refinement covering X.*
   - *Every open cover $\mathcal{U}$ has a locally finite closed refinement covering X.*

4. ***Stone's Theorem:*** *Every metric space is paracompact. If X is a metric space and $\mathcal{U}$ is an open cover of X then there are refinements $\mathcal{V}_n, n \in \mathbb{N}$ of $\mathcal{U}$ such that each $\mathcal{V}_n$ is a discrete family, and $\bigcup_n \mathcal{V}_n$ covers X. I.e. Every open cover of X has an open, σ-discrete refinement covering X.*

5. ***Bing's Metrization Theorem:*** *A space is metrizable if and only if X is perfectly normal and has a σ-discrete basis if and only if X is homeomorphic to a subspace of a countable product of hedgehogs (of some spininess).*

5.3 Proofs

1. If $\mathcal{A}$ is locally finite and $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{A}'$ is still locally finite. It is thus sufficient to show $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$. $\supseteq$ is clear. For $\subseteq$, assume that $x \notin \bigcup_{A \in \mathcal{A}} \overline{A}$. Let $U \ni x$ be open such that U meets only finitely many elements of $\mathcal{A}$, say $A_1, \ldots, A_n$. For each $i = 1, \ldots, n$, $x \notin \overline{A_i}$, so choose open $V_i \ni x$ disjoint from A_i. Then $x \in U \cap \bigcap_i V_i$ and the RHS is open and disjoint from $\bigcup \mathcal{A}$ as required.

2. If C, D are disjoint closed, for each $c \in C$ choose open $U_c \ni c$ such that $\overline{U_c} \cap D = \emptyset$. Then $\{U_c : c \in C\} \cup \{X \setminus C\}$ is an open cover of X so has a locally finite open refinement $\mathcal{V}'$. Let $\mathcal{V} = \{V \in \mathcal{V}' : V \cap C \neq \emptyset\}$, still a locally finite family which refines $\{U_c : c \in C\}$ and covers C. Since locally finite families are closure preserving we have $\overline{\bigcup_{V \in \mathcal{V}} V} = \bigcup_{V \in \mathcal{V}} \overline{V} \subseteq \bigcup_{c \in C} \overline{U_c}$ disjoint from D, so that $U = \bigcup \mathcal{V}$ is as required.

3. Exercise Sheet.

4. **Stone's Theorem:** Suppose $\mathcal{U}$ is an open cover of X. Well order $\mathcal{U}$ by some well-order $\leq$ (using Choice). For each $n \in \mathbb{N}$ and $U \in \mathcal{U}$, define

$$S_{U,n} = \left\{x \in X : B_{3/2^n}(x) \subseteq U \wedge \forall U' < U \ x \notin U'\right\}$$

and let

$$V_{U,n} = \bigcup_{x \in S_{U,n}} B_{1/2^n}(x),$$

an open subset of U. If $y \in V_{U,n}$ and $y' \in V_{U',n}$ with (wlog) $U < U'$ then there is $x \in S_{U,n}$ with $d(x,y) < 1/2^n$ and $x' \in S_{U',n}$ with $d(x',y') < 1/2^n$. But if $x' \in S_{U',n}$ then $x' \notin U$ so $d(x,x') \geq 3/2^n$ and hence $d(y,y') \geq d(x,x') - d(x,y) - d(x',y') \geq 1/2^n$. Thus each $B_{2^{-(n+1)}}(y), y \in X$ meets at most one $V_{U,n}, U \in \mathcal{U}$ and hence $\mathcal{V}_n = \{V_{U,n} : U \in \mathcal{U}\}$ is a discrete family. Clearly $V_{U,n} \subseteq U$. Finally $\bigcup_n \mathcal{V}_n$ covers X, since for $x \in X$, choose $U \in \mathcal{U}$ minimal such that $x \in U$ and as U is open, find $n \in \mathbb{N}$ with $B_{3/2^n}(x) \subseteq U$ giving $x \in V_{U,n}$.

Noting that metric spaces are regular and that σ-discrete implies σ-locally finite, we see that metric spaces are paracompact. (In fact, by defining $S_{U,n} = \left\{x \in X : B_{3/2^n}(x) \subseteq U \wedge \forall U' < U \; x \notin U' \wedge x \notin \bigcup_{U \in \mathcal{U}, n' < n} V_{U,n'}\right\}$ you could directly obtain a locally finite open refinement.)

5. **Bing's Metrization Theorem:** $d_C(x) = \inf \{d(x,c) : c \in C\}$ witnesses perfect normality of metric spaces. For the σ-discrete base, apply Stone's Theorem to each $\{B_{2^{-n}}(x) : x \in X\}$ to obtain a σ-discrete open refinement $\mathcal{V}_n$ covering X and then note that $\bigcup_n \mathcal{V}_n$ is a σ-discrete basis of X.

   Now assume that X is perfectly normal and has a σ-discrete basis $\mathcal{B} = \bigcup_n \mathcal{B}_n$ with each $\mathcal{B}_n$ discrete. Fix $n \in \mathbb{N}$. For each $B \in \mathcal{B}_n$, let $f_B : X \to [0,1]$ be continuous such that $f^{-1}(0) = X \setminus B$ and define $F_n : X \to H_{\mathcal{B}_n}$ by $F(x) = 0$ if $x \notin \bigcup \mathcal{B}_n$ and $F(x) = (f_B(x), B)$ if $x \in B \in \mathcal{B}_n$. This is well-defined since $\mathcal{B}_n$ is discrete (each x is in at most one B). It is continuous since each f_B is continuous and $\mathcal{B}_n$ is discrete: for $x \in X$, choose open $U \ni x$ that meets at most one element of $\mathcal{B}_n$, say B. Then $F_n|_U = f_B$ is continuous on U, hence F_n is continuous at x. Note that $\left\{{F_n}^{-1}(U) : U \text{ open } \subseteq H_{\mathcal{B}_n}\right\} \supseteq \mathcal{B}_n$. Thus $\{F_n : n \in \mathbb{N}\}$ satisfies the conditions of the Embedding Lemma and hence X is homeomorphic to a subspace of a countable product of hedgehogs.

   Finally, a countable product of metric spaces is metrizable.

6 Connectedness, Zero-Dimensionality

6.1 Definitions

A disconnection of X is a partition of X into two non-empty closed-and-open (clopen) subsets. X is disconnected if and only if there is a disconnection of X.

X is connected if and only if it is not disconnected.

The component of a point $x \in X$ is the greatest connected subspace $C(x)$ of X containing x.

The quasicomponent of a point $x \in X$ is $Q(x) = \bigcap \{F \subseteq X : x \in F \text{ clopen}\}$.

X is totally disconnected if and only if every component is a singleton.

X is zero-dimensional if and only if X has a basis of clopen sets.

6.2 Results

Lemma 6.1 (Recall)**.** *1. X is connected if and only if every continuous function into the discrete two-point space is constant.*

2. Suppose $A, A_i \subseteq X, i \in I$ are connected. If for each $i \in I$, $A \cap A_i \neq \emptyset$ then $A \cup \bigcup_i A_i$ is connected.

3. The component of a point exists and equals $\bigcup \{C \subseteq X : x \in C$ connected$\}$.

Theorem 6.2. *1. Both the components and the quasicomponents of a space form a partition.*

2. For every $x \in X$, $C(x) \subseteq Q(x)$.

*3. **Sura-Bura Lemma:** If X is compact Hausdorff, then for every $x \in X$, $C(x) = Q(x)$.*

4. A totally disconnected compact Hausdorff space is zero-dimensional.

6.3 Proofs

3. **Sura-Bura Lemma:** Suppose that some quasicomponent $Q = Q(x)$ is disconnected, i.e. there are Q-clopen non-empty disjoint A, B such $Q = A \cup B$. As Q is closed (an intersection of closed sets), A, B are closed in X. As X is compact Hausdorff, it is normal, so there are disjoint open $U \supseteq A, V \supseteq B$. As X is compact and $\{U \cup V\} \cup \{X \setminus F : x \in F \text{ clopen} \subseteq X\}$ covers X, there are finitely many X-clopen $F_1, \ldots, F_n$ containing x such that $U \cup V, F_1, \ldots, F_n$ covers X so that $x \in F = \bigcap_i F_i \subseteq U \cup V$ and F is clopen (in X). Now $\overline{F \cap U} \subseteq \overline{F} \cap \overline{U} \subseteq F \cap (X \setminus V) \subseteq (F \cap (U \cup V)) \cap (X \setminus V) \subseteq F \cap U$. Hence $F \cap U$ is X-clopen and similarly $F \cap V$ is X-clopen. Wlog $x \in F \cap U$ and hene $Q \subseteq F \cap U$, contradicting $B \neq \emptyset$.