Sheet 2: Schwarz–Christoffel, boundary value problems

Q1 [**Optional**] The domain D in the ζ-plane is bounded by a polygon with exterior angles $\beta_j\pi$, $j = 1, \ldots, n$. The conformal map $\zeta = f(z)$ maps the upper half-plane $\mathrm{Im}(z) > 0$ onto D, the finite points $x_1 < x_2 < \ldots < x_n$ on the real axis being mapped to the vertices of the polygon. Verify the Schwarz–Christoffel formula

$$\frac{\mathrm{d}f}{\mathrm{d}z} = C \prod_{j=1}^{n} (z - x_j)^{-\beta_j},$$

where C is a constant. In general, how many of the x_j can be specified independently? How is the formula modified if $x_n = \infty$?

Q2 Write down, as an integral, the Schwarz–Christoffel map from a half-plane to a rectangle, with the vertices being the images of the points $z = \pm 1$ and $z = \pm a$, where $a > 1$ is real. Explain why a cannot be specified arbitrarily, but is determined by the aspect ratio of the rectangle.

Q3 The domain D consists of the upper half-plane with a solid wall along the real axis. The segment of the imaginary axis from $z = 0$ to $z = \mathrm{i}$ is also impermeable to fluid. Find the complex potential for inviscid incompressible irrotational flow in D with velocity $(U_1, 0)$ at infinity.

Q4 The domain D consists of the right-hand half plane $x > 0$ with the circle $|z - a| = b$, $0 < b < a$, and its interior removed. Find the temperature $u(x, y)$ in steady heat flow if $u = 0$ on the y axis, $u = 1$ on $|z - a| = b$, and $u \to 0$ at infinity.

Q5 (a) Show that the complex potential for uniform flow with unit speed at an angle θ to the real axis is $w(z) = \mathrm{e}^{-\mathrm{i}\theta} z$.

(b) Hence find the potential for flow past the circle $|z| = R$ with the same uniform flow in the far field.

(c) Calculate the potential for flow past the ellipse

$$\frac{x^2}{(R + 1/R)^2} + \frac{y^2}{(R - 1/R)^2} = 1$$

with the same far-field condition.

Q6 (a) Carefully define a branch of the function $\cosh^{-1}(Z)$ that is holomorphic in the upper half-plane. What is $\cosh^{-1}(0)$? What is the derivative of $\cosh^{-1}(Z)$?

(b) Show that the Schwarz–Christoffel map from the upper half-plane to the exterior of the half-strip $0 < x < \infty$, $-1 < y < 1$ has the form

$$z = A + C\left(Z\sqrt{Z^2 - 1} - \cosh^{-1} Z\right),$$

and find the constants A and C.

(c) Hence find the complex potential $w(z)$ for potential flow past this obstacle with a uniform stream $(U_1, 0)$ at infinity.

<u>Hints</u> Q4: Show that the mapping $\zeta = (z - \alpha)/(z + \alpha)$, with α real and positive, takes D onto an annular region with the imaginary axis mapping to $|\zeta| = 1$ and show that, if $\alpha^2 = a^2 - b^2$, then the image of D is a concentric circular annulus. Q5(c): Consider the inverse of the Joukowski map. Q6(b): Map $Z = \pm 1$ to the finite corners of the domain, and $Z = \infty$ to the vertex at $x = \infty$. Q6(c): Bearing in mind the behaviour of the mapping at infinity, think carefully about the potential in the Z plane: it is not 'constant $\times$ Z.'