Sheet 3: Free surface flows

Q1 Inviscid irrotational fluid flows steadily in the domain Ω shown in figure 1, between a rigid wall ABC consisting of two semi-infinite straight line segments meeting at right angles, and a free surface $A'C'$.

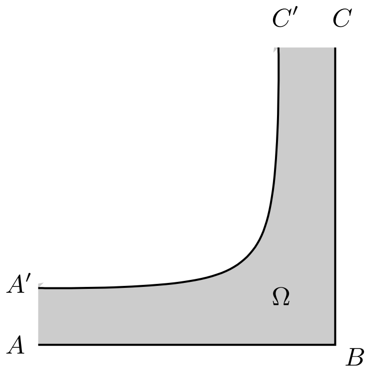

Figure 1: A jet climbing a wall.

The fluid layer has thickness 1 and velocity $(1,0)$ far upstream, at AA'. The boundary value problem for the complex potential $w(z) = \phi + \mathrm{i}\psi$ is that $w(z)$ is holomorphic in Ω, with

$$\psi = 0 \text{ on } ABC, \qquad \psi = 1, \quad |w'| = 1 \text{ on } A'C',$$

where $w'(z) = u - \mathrm{i}v$ is the complex velocity. In addition, take the reference point for ϕ so that $w = 0$ at B.

(a) Show that flow domain in the potential plane (w) is a strip, while in the hodograph plane (w') it is a quarter of the unit circle.

(b) Show that the map to a half plane is

$$\zeta = e^{\pi w} = \left(\frac{(w')^2 - 1}{(w')^2 + 1}\right)^2.$$

(c) Parametrise the free surface $A'C'$ by $w' = \mathrm{e}^{-\mathrm{i}\theta}$, where $0 \le \theta \le \pi/2$. Show that

$$\zeta = -\tan^2\theta, \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = \frac{2}{\pi}(\operatorname{cosec}\theta + \mathrm{i}\sec\theta) \qquad \text{on} \quad A'C'.$$

(d) Find parametric equations for the free surface from the real and imaginary parts of $\mathrm{d}z/\mathrm{d}\theta$. Check that it looks as it should.

Q2 A two-dimensional jet of inviscid irrotational fluid, of thickness $2h_\infty$ and moving to the right with speed 1, enters a semi-infinite rectangular cavity with walls at $y = \pm 1$ and $x = L$, as shown in figure 2; the y axis is tangent to the free surface.

The boundary value problem for the complex potential $w(z) = \phi + \mathrm{i}\psi$ for the upper half of the flow (within the strip $0 < y < 1$, $-\infty < x < L$) is that $w(z)$ is holomorphic in the fluid region, with

$$\psi = 0 \text{ on } ABCDE, \qquad \psi = h_\infty, \quad |w'| = 1 \text{ on } A'E',$$

where $w'(z) = u - \mathrm{i}v$ is the complex velocity. In addition, take the reference point for ϕ so that $w = 0$ at C.

(a) Sketch the flow domain in the potential and hodograph planes.

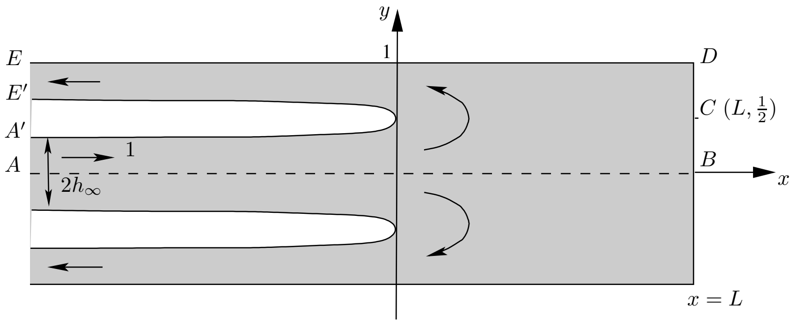

Figure 2: A jet entering a box.

(b) Now consider the case $L = \infty$, with stagnant fluid far inside the cavity.

(i) Show that B, C and D coincide at the origin in the hodograph plane, so that the flow domain is the whole interior of the semicircle in the lower half plane.

(ii) Show that

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1 - \mathrm{e}^{\pi w/2h_\infty}}{1 + \mathrm{e}^{\pi w/2h_\infty}} = -\tanh\frac{\pi w}{4h_\infty}.$$

Find w satisfying $w = \mathrm{i}h_\infty$ at $z = \mathrm{i}/2$, the tip of the air finger shown in figure 2.

(iii) Show that the free surface for this flow, $w = \phi + \mathrm{i}h_\infty$, $-\infty < \phi < \infty$, satisfies

$$\mathrm{e}^{-\pi x/2h_\infty} \cos\left(\frac{\pi(y - \frac{1}{2})}{2h_\infty}\right) = 1,$$

and show that $y \to \pm h_\infty$ as $x \to -\infty$ is only consistent if h_∞ takes a particular value, which you should find.

Q3 **[Optional]** Consider potential flow with a free surface (and no gravity) over a horizontal base along the x axis, with a thin vertical obstacle along the y axis from $z = 0$ to $z = \mathrm{i}$. The flow far away from the obstacle is uniform with velocity $(1, 0)$ and height h_∞, and the free surface is assumed to be symmetric about $x = 0$.

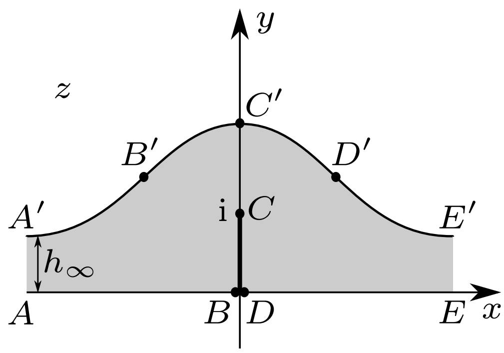

Figure 3: Free surface flow over an obstacle.

Taking the point C to be at $w = 0$ and B and D at $w = \pm\beta$, construct the flow domain in the potential and hodograph planes. Show that the transformation $\zeta = 2/(1 - w'^2)$ maps the latter onto a polygonal domain, and the transformation $\zeta_2 = \tanh(\pi w/2h_\infty)$ maps the potential plane to the upper half plane.

Hence use a Schwarz–Christoffel transformation to find an implicit relationship between w' and w, depending on β (in principle β can be related to h_∞ once $w' = F(w)$ is determined).