Sheet 5: Applications of the Plemelj formulae, transforms

Q1 Suppose f satisfies the Cauchy singular integral equation

$$a(t)f(t) + \frac{b(t)}{\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} = c(t) \qquad \text{on} \quad \Gamma, \qquad (*)$$

where a, b and c are holomorphic in a neighbourhood of Γ.

(a) Show that, if

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

then $(a+b)w_+ + (b-a)w_- = c$ on Γ.

(b) Now suppose $a+b$ and $a-b$ are not zero on Γ, and that $\tilde{w}$ is holomorphic and non-zero away from Γ and that $(a+b)\tilde{w}_+ = -(b-a)\tilde{w}_- \neq 0$ on Γ. Show that

$$\left(\frac{w}{\tilde{w}}\right)_+ - \left(\frac{w}{\tilde{w}}\right)_- = \frac{c}{(a+b)\tilde{w}_+} \qquad \text{on} \quad \Gamma.$$

(c) Hence show that

$$w(z) = \frac{\tilde{w}(z)}{2\pi \mathrm{i}} \int_\Gamma \frac{c(\zeta)\,\mathrm{d}\zeta}{(a(\zeta)+b(\zeta))\tilde{w}_+(\zeta)(\zeta - z)}$$

is a possible solution for $w(z)$ and that $(*)$ is satisfied by

$$f(t) = -\frac{b(t)\tilde{w}_+(t)}{(a(t)-b(t))} \frac{1}{\pi \mathrm{i}} \fint_\Gamma \frac{c(\zeta)}{(a(\zeta)+b(\zeta))\tilde{w}_+(\zeta)} \frac{\mathrm{d}\zeta}{\zeta - t} + \frac{c(t)a(t)}{a(t)^2 - b(t)^2}.$$

Q2 Suppose $f(x) = \mathrm{e}^{|x|}$ for $-\infty < x < \infty$.

(a) If $f(x) = f_+(x) + f_-(x)$, where $f_+(x) = 0$ for $x < 0$ and $f_-(x) = 0$ for $x > 0$, show that the Fourier transforms of f_+ and f_- are given by

$$\bar{f}_+(k) = \int_{-\infty}^{\infty} f_+(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = \frac{\mathrm{i}}{k - \mathrm{i}} \qquad \text{for} \quad \mathrm{Im}(k) > 1$$

and

$$\bar{f}_-(k) = \int_{-\infty}^{\infty} f_-(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = -\frac{\mathrm{i}}{k + \mathrm{i}} \qquad \text{for} \quad \mathrm{Im}(k) < -1$$

To which parts of the complex k-plane may $\bar{f}_\pm(k)$ be analytically continued?

(b) Use contour integration to evaluate

$$\frac{1}{2\pi} \fint_{-\infty + \mathrm{i}\alpha}^{\infty + \mathrm{i}\alpha} \bar{f}_+(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k \quad \text{and} \quad \frac{1}{2\pi} \fint_{-\infty + \mathrm{i}\beta}^{\infty + \mathrm{i}\beta} \bar{f}_-(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k$$

for $x < 0$ and $x > 0$, where $\alpha > 1$ and $\beta < -1$.

(c) Over which part of the complex k-plane is it possible to define $\bar{f}(k)$? Sketch a suitable inversion contour Γ for which

$$f(x) = \frac{1}{2\pi} \fint_\Gamma \bar{f}(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k.$$

Verify this result using contour integration.

Q3 (a) Show that

$$w(z) = \int_\Gamma g(\zeta) \mathrm{e}^{z\zeta} \, \mathrm{d}\zeta \qquad (\dagger)$$

is a solution of Airy's equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + zw = 0$$

only if $g(\zeta) = A\mathrm{e}^{\zeta^3/3}$ and $[g(\zeta)\mathrm{e}^{z\zeta}]_\Gamma = 0$, where A is a constant. Identify two choices for Γ which lead to two independent solutions of the differential equation.

(b) Show that $(\dagger)$ is a solution of Bessel's equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w = 0$$

only if $g(\zeta) = A/(1+\zeta^2)^{1/2}$ and $[(1+\zeta^2)g(\zeta)\mathrm{e}^{z\zeta}]_\Gamma = 0$. Identify two choices for Γ which lead to two independent solutions of the differential equation for $\mathrm{Re}(z) > 0$.