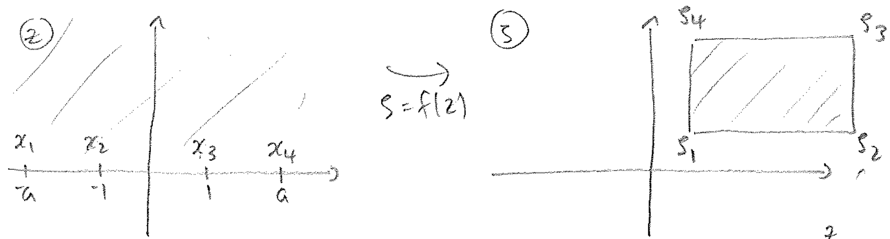
2.



$\beta = \frac{1}{2}$ at each vertex

Schwarz-Christoffel formula $\Rightarrow$ $f(z) = A + C\int^z (t-1)^{-1/2}(t+1)^{-1/2}(t-a)^{-1/2}(t+a)^{-1/2}\,dt$

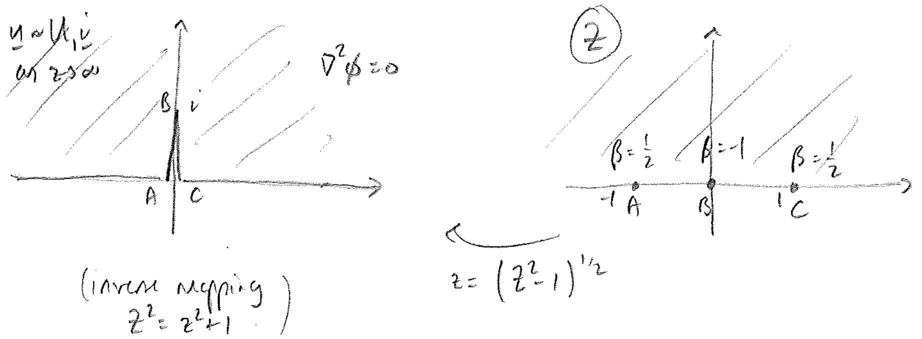
$$= A + C\int^z \frac{dt}{(t^2-1)^{1/2}(t^2-a^2)^{1/2}}$$

A and C only control the location, orientation and scaling of the whole rectangle, so the aspect ratio must be controlled by a (i.e. there is nothing else that could control it).

Note the lengths of the sides are $l_1 = |f(1) - f(-1)|$ and $l_2 = |f(a) - f(1)|$, which are

$$l_1 = |C|\left|\int_{-1}^{1} \frac{dt}{(t^2-1)^{1/2}(t^2-a^2)^{1/2}}\right| \quad l_2 = |C|\left|\int_{1}^{a} \frac{dt}{(t^2-1)^{1/2}(t^2-a^2)^{1/2}}\right| \quad \text{so} \quad \frac{l_1}{l_2} = \left|\frac{\int_{-1}^{1} \frac{dt}{(t^2-1)^{1/2}(t^2-a^2)^{1/2}}}{\int_{1}^{a} \frac{dt}{(t^2-1)^{1/2}(t^2-a^2)^{1/2}}}\right|$$

3.



Schwarz Christoffel.

$$z = f(Z) = A + C\int^Z \frac{t\,dt}{(t^2-1)^{1/2}}$$

$$= A + C(Z^2-1)^{1/2}$$

$1 \mapsto 0 \Rightarrow A = 0$; $0 \mapsto i \Rightarrow C = 1$

so $z = (Z^2-1)^{1/2}$

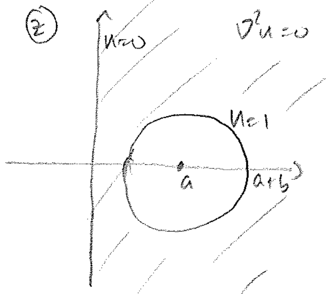
We'd like to find complex potential $w(z)$ satisfying $\boxed{w \sim U_1 z \text{ as } z \to \infty, \quad \operatorname{Im} w = 0 \text{ on boundary}}$ (with suitable choice of branch)

Converting to Z plane, noting $z \sim Z$ for large z, we want $W(Z)$ with $W \sim U_1 Z$ as $Z \to \infty$, & $\operatorname{Im} W = 0$ on boundary, which is now just the real axis. So $W = U_1 Z$ works everywhere.

Hence $\boxed{w(z) = W(Z) = U_1(z^2+1)^{1/2}}$

(Note we could work out from this the velocity potential and stream function, since $w = \phi + i\psi$).

4. (z)



Want to map D to an annulus. (There's no way we could map both boundaries to straight lines, since in that case they would 'touch' at ∞, and they clearly don't touch. If at least one of them has to remain circular, then it would obviously be easier if they are both circular, and concentric, so we could use polar coordinates in mapped domain)

- Consider mapping $s = \frac{z-\alpha}{z+\alpha}$, which, since Möbius, maps the boundaries to circles.
- For $z = iy$, $s = \frac{-(\alpha - iy)}{\alpha + iy}$ which has unit modulus, so the imaginary axis maps to the unit circle.
- For $z = a + be^{i\theta}$, $s = \frac{be^{i\theta} - (\alpha - a)}{be^{i\theta} + (\alpha + a)}$. If this is to map to a concentric circle $|s| = R$, we need

$$s\bar{s} = R^2 = \frac{(be^{i\theta} - (\alpha-a))(be^{-i\theta} - (\alpha-a))}{(be^{i\theta} + (\alpha+a))(be^{-i\theta} + (\alpha+a))} = \frac{b^2 - 2(\alpha-a)\cos\theta + (\alpha-a)^2}{b^2 + 2(\alpha+a)\cos\theta + (\alpha+a)^2} = -\frac{(\alpha-a)}{(\alpha+a)} + \frac{b^2 + \frac{\alpha-a}{\alpha+a}b^2 + (\alpha-a)^2 + \frac{\alpha-a}{\alpha+a}(\alpha+a)^2}{b^2 + 2(\alpha+a)\cos\theta + (\alpha+a)^2}$$

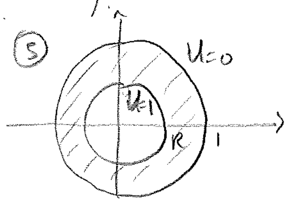
& to ensure this quantity is independent of θ, we must have the last numerator equal to zero.

i.e. $2\alpha b^2 + 2\alpha(\alpha^2 - a^2) = 0 \quad \Rightarrow$ we need $\boxed{\alpha^2 = a^2 - b^2}$.

So with this choice of α, the circle $|z-a| = b$ maps to the circle $\boxed{|s| = R = \left(\frac{a-\alpha}{a+\alpha}\right)^{1/2}} < 1$

(Note $R = \frac{(a-\alpha)}{b} = \frac{a - \sqrt{a^2-b^2}}{b}$).

- So $s = \frac{z-\alpha}{z+\alpha}$ maps to



Solution of Laplace in polar coordinates $\left(\nabla^2 u = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0\right)$ are $u = A\log r + B$.

The solution here has $U = \frac{\log r}{\log R}$ (satisfies $U = 0$ on $r = 1$, $U = 1$ on $r = R$)

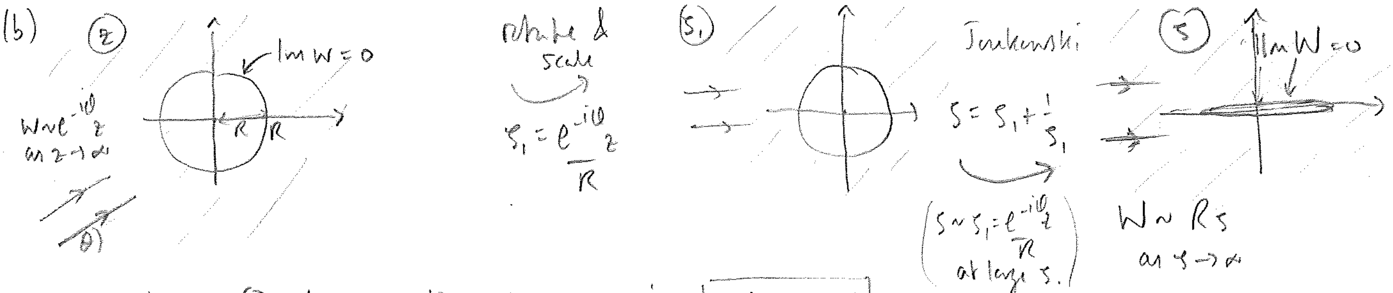
$$= \mathrm{Re}\left(\frac{\log s}{\log R}\right)$$

Hence $\boxed{u = \mathrm{Re}\left(\frac{\log\left(\frac{z-\alpha}{z+\alpha}\right)}{\log\left(\frac{a-\alpha}{a+\alpha}\right)^{1/2}}\right) = \frac{\log\left|\frac{z - \sqrt{a^2-b^2}}{z + \sqrt{a^2-b^2}}\right|}{\log\left|\frac{a - \sqrt{a^2-b^2}}{b}\right|}}$

5. (a) $\underline{u} = (\cos\theta, \sin\theta)$ has $\frac{dw}{dz} = u - iv = \cos\theta - i\sin\theta = e^{-i\theta}$ for the complex velocity.

Hence $w = e^{-i\theta} z$.

(b)



Clearly in the (ζ) domain the solution is just $\boxed{W(\zeta) = R\zeta.}$

Hence the complex potential in the (z) plane is $w(z) = R\left(\frac{e^{-i\theta}z}{R} + \frac{R}{e^{-i\theta}z}\right) = \boxed{e^{-i\theta}z + \frac{R^2 e^{i\theta}}{z}}$

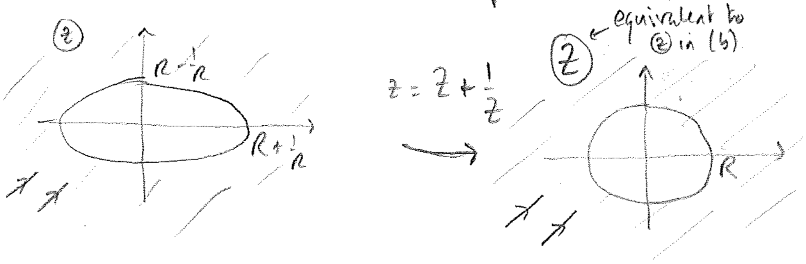
(c) Note that the Joukowski map takes circles to ellipses, i.e. if $Z = Re^{i\theta}$, then

$$z = Z + \frac{1}{Z} = Re^{i\theta} + \frac{1}{R}e^{-i\theta} = \left(R + \frac{1}{R}\right)\cos\theta + i\left(R - \frac{1}{R}\right)\sin\theta, \text{ which has } \frac{x^2}{\left(R+\frac{1}{R}\right)^2} + \frac{y^2}{\left(R-\frac{1}{R}\right)^2} = 1$$

So the inverse mapping $\left(Z^2 - zZ + 1 = 0 \Rightarrow \left(Z - \tfrac{1}{2}z\right)^2 = \tfrac{1}{4}z^2 - 1 \Rightarrow Z = \tfrac{1}{2}z + \tfrac{1}{2}(z^2-4)^{1/2}\right)$ takes the ellipse to a circle of radius R in the Z plane i.e. to the problem just solved in (b).

(The far field is left unchanged so the far field condition remains the same.)

Hence the solution for flow past the ellipse is $w(z) = \underset{\substack{\uparrow \\ \text{from (b)}}}{W(Z)} = \boxed{e^{-i\theta}\frac{\left(z + (z^2-4)^{1/2}\right)}{2} + \frac{2R^2 e^{i\theta}}{z + (z^2-4)^{1/2}}}$



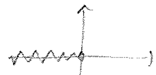
6. (a) $z=\cosh^{-1}(Z) \iff Z=\cosh z = \frac{1}{2}(e^{z}+e^{-z}) \iff e^{2z}-2Ze^{z}+1=0 \iff (e^{z}-Z)^2 = Z^2-1$

$\iff e^{z} = Z+(Z^2-1)^{1/2} \iff z=\log\left(Z+(Z^2-1)^{1/2}\right)$

Define $(Z^2-1)^{1/2} = |Z^2-1|^{1/2} e^{i(\theta_1+\theta_2)/2}$ where $\theta_1=\arg(Z-1)\in(-\pi,\pi)$

$\theta_2=\arg(Z+1)\in[-\pi,\pi)$

and take $\log(\zeta)=\log|\zeta|+i\theta$ where $\theta=\arg(\zeta)\in[-\pi,\pi)$

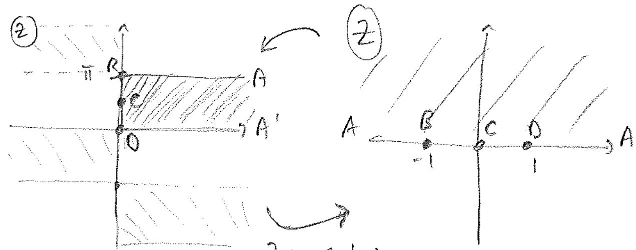


Then $\boxed{\cosh^{-1}(Z)=\log\left(Z+(Z^2-1)^{1/2}\right)}$ is holomorphic in the upper half plane.

With this definition, $Z=0$ has $\theta_1=\pi$, $\theta_2=0$ so $(Z^2-1)^{1/2}=e^{i\pi/2}=i$, so $Z+(Z^2-1)^{1/2}=i$, which has $\theta=\frac{\pi}{2}$.

Hence $\cosh^{-1}(0)=\frac{i\pi}{2}$.

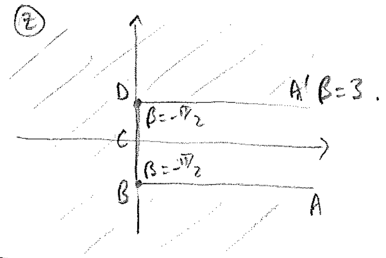
$$\frac{d}{dZ}\cosh^{-1}(Z) = \frac{1+\frac{Z}{(Z^2-1)^{1/2}}}{Z+(Z^2-1)^{1/2}} = \frac{1}{(Z^2-1)^{1/2}}.$$

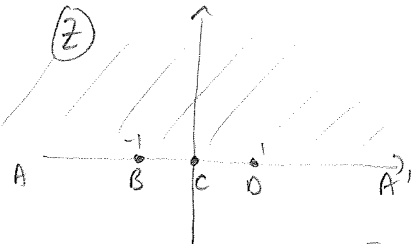


$Z=\cosh(z)$
$=\cosh x\cos y - i\sinh x \sin y$

- Note, the branch can also be 'defined' graphically. There are many regions of the z plane that map to the upper half Z plane under $Z=\cosh z$. The choice of branch is equivalent to deciding which of these we want to map back to.

(b)





$$z=A+C\int^{Z}(t^2-1)^{1/2}\,dt$$

$$=A+C\,\frac{1}{2}\left(Z(Z^2-1)^{1/2}-\cosh^{-1}Z\right)$$

using the result below, and (a)

[Note $\int(t^2-1)^{1/2}\,dt = \int\frac{t^2}{(t^2-1)^{1/2}}\,dt-\int\frac{dt}{(t^2-1)^{1/2}}$

$= t(t^2-1)^{1/2}-\int(t^2-1)^{1/2}\,dt-\int\frac{dt}{(t^2-1)^{1/2}}$]

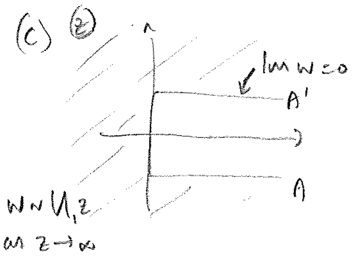
To map C to C, take $A+C\left(-\frac{i\pi}{4}\right)=0$

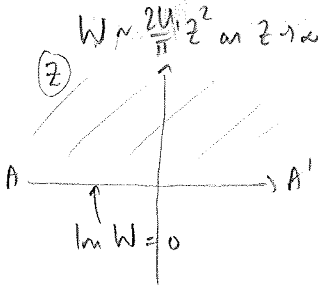
D to D, take $A=i$

$\Rightarrow C=\frac{4}{\pi}$

so $\boxed{z=i+\frac{2}{\pi}\left(Z(Z^2-1)^{1/2}-\cosh^{-1}Z\right)}$

(c)





For large Z, $\cosh^{-1}Z$ is only logarithmic, so

$z\sim\frac{2}{\pi}Z^2$.

Hence $w\sim U_1 z \Rightarrow W\sim U_1\frac{2}{\pi}Z^2$ for far field.

By inspection $\boxed{W=\frac{2U_1}{\pi}Z^2}$ satisfies the required conditions

Hence $\boxed{w(z)=W(Z(z))}$ this inverse mapping does not look like it has a nice form.

We could write implicitly as $z=i+\frac{2}{\pi}\left[\left(\frac{\pi W}{2U_1}\right)^{1/2}\left(\frac{\pi W}{2U_1}-1\right)^{1/2}-\cosh^{-1}\left(\frac{\pi W}{2U_1}\right)\right]$