Noncommutative Rings, HT 2018
Problem Sheet 5

Throughout this sheet, A will denote a ring.

1. Show that a maximal two-sided ideal in A is left primitive, and a left primitive ideal in A is prime. Find an example of a ring A, and a prime ideal P in A, such that P is not left primitive.

2. (a) Let $A = k[x, y, z]$ be the polynomial ring in three variables over a field k, and let $I = (xy, yz, zx)$. Find $\min(I)$, and justify your answer.

   (b) Suppose that A is a commutative Noetherian graded ring, and let I be a graded ideal in A. Prove that $\sqrt{I}$ is also a graded ideal.

3. Suppose that A is commutative and Noetherian.

   (a) If M is a finitely generated A-module and $I = \operatorname{Ann}_A(M)$, show that A/I is isomorphic to an A-submodule of M^n for some $n \in \mathbb{N}$.

   (b) If $J \lhd A$ and d is a dimension function for A, prove that $d(A/J) = d(A/J^m)$ for all $m \geq 1$.

   (c) Prove that a dimension function for A is completely determined by the values it takes on modules of the form A/P where $P \in \operatorname{Spec}(A)$.

4. Let $A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and let $P = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Show that P is a prime ideal in A. Also, show that $S := A \backslash P$ is multiplicatively closed but is not a right Ore set. Prove that S *is* a left localisable subset of A and that $S^{-1}A \cong \mathbb{Q}$.

5. Suppose that A is left Noetherian, and let S be a left localisable subset of A.

   (a) Prove that $Q := S^{-1}A$ is also left Noetherian.

   (b) Show that if I is a two-sided ideal in A then $Q \cdot I$ is also a two-sided ideal in Q.

   (c) Suppose further that A is also right Noetherian, and that P is a prime ideal in A such that $P \cap S = \emptyset$. Show that $Q \cdot P$ is a prime ideal in Q.

6. Suppose that A is commutative, and write $A_P := (A \backslash P)^{-1}A$ for every $P \in \operatorname{Spec}(A)$.

   (a) Suppose that A_P has no non-zero nilpotent elements for all $P \in \operatorname{Spec}(A)$. Show that A also has no non-zero nilpotent elements.

   (b) If A_P is an integral domain for all $P \in \operatorname{Spec}(A)$, must A be an integral domain, too?