Noncommutative Rings, HT 2018
Problem Sheet 6

1. Let A be a filtered ring and let M be a filtered left A-module.
   (a) Show that $\widetilde{M}/t\widetilde{M}$ is isomorphic to $\operatorname{gr} M$ as a left $\operatorname{gr} A$-module.
   (b) Viewing M as a left $\widetilde{A}$-module via the isomorphism $\widetilde{A}/(t-1)\widetilde{A} \cong A$ from Lemma 4.20(2), show that $\widetilde{M}/(t-1)\widetilde{M}$ is isomorphic to M as a left $\widetilde{A}$-module.

2. (a) Verify that the commutator bracket on a ring A is a Poisson bracket.
   (b) Let k be a field. Suppose that $\{,\}$ is a Poisson bracket on the polynomial ring $A = k[x_1, \ldots, x_n]$ such that $\{k, A\} = 0$. Prove that $\{,\}$ is completely determined by its values on the x_i's.
   (c) Let A be a filtered ring such that $\operatorname{gr} A$ is commutative, and let $\{,\}$ be the induced Poisson bracket on $\operatorname{gr} A$. Show that $\operatorname{gr} I$ is closed under $\{,\}$ for any left ideal I in A.
   (d) Find an example of a filtered ring A and a graded ideal J in $\operatorname{gr} A$ such that $\operatorname{gr} A$ is commutative and $\{J, J\} \subseteq J$ but $\{\sqrt{J}, \sqrt{J}\} \nsubseteq \sqrt{J}$.

3. Let B be a left Noetherian ring, and let $t \in B$ be a central regular element. By considering the ring $(t^{\mathbb{N}})^{-1}B$ or otherwise, show that for any left ideal I of B there is an integer n such that $I \cap t^n B \subseteq tI$.

4. Let $n \geq 1$, and let k be a field of characteristic zero. Show that there are no $n \times n$ matrices X, Y with entries in k that satisfy the relation $YX - XY = 1$. What happens if the characteristic of k is positive?

5. Let R be a filtered ring, let M be a filtered left $R-$module with filtration $(M_i)_{i \in \mathbb{Z}}$ and let N be a submodule of M. Equip N with the *subspace filtration* $N_i := N \cap M_i$, and equip M/N with the *quotient filtration* $(M/N)_i := (M_i + N)/N$. Show that
   (a) there is an injective $\operatorname{gr} R$-module homomorphism $\alpha : \operatorname{gr} N \to \operatorname{gr} M$,
   (b) there is a surjective $\operatorname{gr} R$-module homomorphism $\beta : \operatorname{gr} M \to \operatorname{gr}(M/N)$,
   (c) $\ker \beta = \operatorname{Im} \alpha$.

6. Let $A = A_n(k)$ be the Weyl algebra, and let r be an integer such that $n \leqslant r \leqslant 2n$. Give an example of a cyclic A-module M such that $d(M) = r$. Justify your answer.

P.T.O.

7. (For the Easter break and the enthusiasts.)

Let A be a filtered ring and let M be a filtered left A-module.

(a) Suppose that $M_n = \{0\}$ for all sufficiently small $n \in \mathbb{Z}$. Show that the filtration on M is good if and only if $\operatorname{gr} M$ is finitely generated as a left $\operatorname{gr} A$-module.

(b) Let $B = \bigoplus_{j \in \mathbb{Z}} B_j$ be a $\mathbb{Z}$-graded ring, and let $F_i B := \bigoplus_{j \leqslant i} B_j$. Show that $(F_i B)_{i \in \mathbb{Z}}$ is a ring filtration on B, and that $\operatorname{gr} B$ is isomorphic to B.

(c) Now suppose that the filtration on A is positive and $\operatorname{gr} A$ is left Noetherian. Show that every graded left ideal of $\widetilde{A}$ is finitely generated, and deduce that $\widetilde{A}$ is also left Noetherian.