2019 | B5.4 | Q1

(a) Euler equations: $\boxed{\rho_t + \nabla\cdot(\rho \underline{u}) \underset{1}{=} 0, \quad \rho\left(\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}\right) \underset{2}{=} -\nabla p.}$

Perturb $\rho = \rho_0 + \rho'$, $p = P(\rho_0) + p'$, $\underline{u} = \underline{u}'$, where primed variables are small.

$$1 \Rightarrow (\rho_0 + \rho')_t + \nabla\cdot(\rho_0 \underline{u}') + \nabla\cdot(\rho' \underline{u}') = 0$$

$$\Rightarrow \boxed{\rho'_t + \rho_0 \nabla\cdot\underline{u}' \underset{3}{=} 0} \quad \text{linearizing.}$$

$$2 \Rightarrow \rho_0 \underline{u}'_t + \rho' \underline{u}'_t + (\rho_0 + \rho')(\underline{u}'\cdot\nabla)\underline{u}' = -\nabla p'$$

$$\Rightarrow \boxed{\rho_0 \underline{u}'_t \underset{4}{=} -\nabla p'} \quad \text{linearizing.}$$

$$p = P(\rho_0) \Rightarrow P(\rho_0) + p' = P(\rho_0 + \rho') = P(\rho_0) + \rho' c_0^2 + \dots$$

by Taylor's Thm with $c_0 = \left(\frac{dP}{d\rho}(\rho_0)\right)^{\frac{1}{2}}$

[B4] $$\Rightarrow \boxed{p' \underset{5}{=} c_0^2 \rho'} \quad \text{linearizing.}$$

$$\nabla\wedge 4 \Rightarrow (\nabla\wedge\underline{u}')_t = -\frac{1}{\rho_0}\nabla\wedge\nabla p' = 0$$

$$\Rightarrow \nabla\wedge\underline{u}' = 0 \text{ for } t \geq 0 \text{ since flow irrotational initially}$$

$$\Rightarrow \exists \phi \text{ s.t. } \boxed{\underline{u}' \underset{6}{=} \nabla\phi} \text{ by hint.}$$

$$4 \,\&\, 6 \Rightarrow \nabla(p' + \rho_0 \phi_t) = 0$$

$$\Rightarrow \boxed{p' + \rho_0 \phi_t = F(t) \underset{7}{=} 0} \quad \text{(wlog absorbing arb. fn F into } \phi)$$

$$\text{Hence,} \quad \nabla^2\phi = -\frac{1}{\rho_0}\rho'_t \qquad (\text{by } 3 \,\&\, 6)$$

$$= -\frac{1}{\rho_0 c_0^2} p'_t \qquad (\text{by } 5)$$

$$= -\frac{1}{\rho_0 c_0^2}(-\rho_0 \phi_t)_t \qquad (\text{by } 7)$$

[B4] giving the wave equation $\boxed{\phi_{tt} = c_0^2 \nabla^2\phi.}$ □

[8]

(b) BCs: $\phi_x = 0$ on $x = 0, a$ for $0 < y < b$, $0 < z < h$;
$\phi_y = 0$ on $y = 0, b$ for $0 < x < a$, $0 < z < h$;
[B1] $\phi_z = 0$ on $z = 0, h$ for $0 < x < a$, $0 < y < b$.

Seek nontrivial separable solution $\phi = e^{-i\omega t} F(x) G(y) H(z)$, then

$$-\frac{\omega^2}{c_0^2} = \underbrace{\frac{F''}{F}}_{-\lambda^2} + \underbrace{\frac{G''}{G}}_{-\mu^2} + \underbrace{\frac{H''}{H}}_{-\nu^2} \qquad (FGH \neq 0)$$

for $\lambda, \mu, \nu \in \mathbb{R}$ since these quantities must be ind. x, y, z, and signs chosen for nontrivial solutions as BCs $\Rightarrow$
$F'(0) = F'(a) = 0$, $G'(0) = G'(b) = 0$, $H'(0) = H'(h) = 0$.

[S3]

For $\lambda = 0$, $F = $ const, but for $\lambda \neq 0$, $F = A\cos(\lambda x) + B\sin(\lambda x)$ ($A, B \in \mathbb{C}$ arb.), so BCs $\Rightarrow$ $B = 0$ and $\lambda a = m\pi$, $m \in \mathbb{Z} \setminus \{0\}$.

Combo $\Rightarrow$ $F \propto \cos\left(\frac{m\pi x}{a}\right)$, $\lambda = \frac{m\pi}{a}$, $m \in \mathbb{Z}$.

Similarly for G and H with $\mu = \frac{n\pi}{b}$, $\nu = \frac{k\pi}{h}$, $n, k \in \mathbb{Z}$.

Combo $\Rightarrow$ $\phi = C e^{-i\omega t} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{k\pi z}{h}\right)$ ($C \in \mathbb{C}$)

are normal modes with natural frequencies ω s.t.

$$\omega^2 = c_0^2(\lambda^2 + \mu^2 + \nu^2) = \pi^2 c_0^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{k^2}{h^2}\right),$$

[S3] where m, n and k are integers. □

$\boxed{7}$

(c) To satisfy no-flux BCs on walls in $z > 0$ and on wavemaker at $z = 0$, seek separable solution

$$\phi = e^{-i\omega t} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) f(z)$$

$$\Rightarrow -\omega^2 f = c_0^2\left(-\frac{m^2\pi^2}{a^2} f - \frac{n^2\pi^2}{b^2} f + f''\right).$$

[S/N2] $\Rightarrow$ $\boxed{f'' + \frac{\Omega^2 - \omega_{m,n}^2}{c_0^2} f = 0 \text{ for } z > 0, \text{ with } f'(0) = V}$

(ii) $\Omega > \omega_{m,n}$ $\Rightarrow$ $f = Ae^{i\alpha z} + Be^{-i\alpha z}$ $(A, B \in \mathbb{C}$ arb.$)$, $\alpha = \left(\frac{\Omega^2 - \omega_{m,n}^2}{c_0^2}\right)^{1/2}$

$$\Rightarrow \phi = \Big(\underbrace{Ae^{i(\alpha z - \Omega t)}}_{\text{Outward}} + \underbrace{Be^{-i(\alpha z + \Omega t)}}_{\text{Inward}}\Big)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Radiation condition $\Rightarrow$ no inward travelling waves as $z \to \infty$ so $B = 0$. Then $f'(0) = V \Rightarrow i\alpha A = V$, giving

[S/N2] $$\boxed{\phi = \frac{V}{i\alpha} e^{i(\alpha z - \Omega t)} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}.$$

swap!

(i) $0 < \Omega < \omega_{m,n}$ $\Rightarrow$ $f = Ae^{-\beta z} + Be^{\beta z}$ $(A, B \in \mathbb{C}$ arb$)$, $\beta = \left(\frac{\omega_{m,n}^2 - \Omega^2}{c_0^2}\right)^{1/2}$

ϕ bdd as $z \to \infty$ $\Rightarrow$ $B = 0$. Then $f'(0) = V \Rightarrow -\beta A = V$, giving

[S/N2] $$\boxed{\phi = -\frac{V}{\beta} e^{-\beta z - i\Omega t} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}$$ $\square$

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(d) Hint $\Rightarrow$ $\boxed{\phi_z(x, y, 0, t) = \frac{qVe^{-i\Omega t}}{\pi^8} \sum_{m,n=1}^{\infty} \frac{\cos\left(\frac{2m\pi x}{a}\right)\cos\left(\frac{2n\pi y}{b}\right)}{m^4 n^4}}$

for $0 < x < a$, $0 < y < b$, so superimposing the solutions in part (c) gives

$$\boxed{\phi = \frac{qVe^{-i\Omega t}}{\pi^8} \sum_{m,n=1}^{\infty} F_{2m,2n}(z) \frac{\cos\left(\frac{2m\pi x}{a}\right)\cos\left(\frac{2n\pi y}{b}\right)}{m^4 n^4}},$$

where

$$F_{m,n}(z) = \frac{c_0}{i(\Omega^2 - \omega_{m,n}^2)^{1/2}} e^{i(\Omega^2 - \omega_{m,n}^2)^{1/2} z / c_0} \quad \text{for } \omega_{m,n} < \Omega,$$

[N4] $$F_{m,n}(z) = -\frac{c_0}{(\omega_{m,n}^2 - \Omega^2)^{1/2}} e^{-(\omega_{m,n}^2 - \Omega^2)^{1/2} z / c_0} \quad \text{for } \omega_{m,n} > \Omega.$$

4 $\square$

2019 | B5.4 | Q2

(a) Constant density so $\nabla \cdot \underline{u} = 0$ and velocity potential is ϕ so $\underline{u} = \nabla \phi$, so $\nabla^2 \phi = 0$ in $z < 0$ after

[B2] linearizing for small η.

Since $\phi_z(x, \eta, t) = \phi_z(x, 0, t) + \eta \phi_{zz}(x, 0, t) + \text{h.o.t.}$ by Taylor's Theorem for small η, linearizing and

[B2] imposing KBC on $z = 0$ $\Rightarrow$ $\phi_z = \eta_t$ on $z = 0$.

Bernoulli's equation for irrotational flow states $\phi_t + \frac{1}{2}|\nabla\phi|^2 + gz + p/\rho = F(t)$ an arbitrary fn of t. Set $F(t) = 0$ wlog, then DBC becomes

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta + \frac{\varepsilon p_a}{\pi\rho(x^2+\varepsilon^2)} = 0 \text{ on } z = \eta.$$

[S|N2] Linearize and impose on $z = 0 \Rightarrow \phi_t + g\eta = -\frac{\varepsilon p_a}{\pi\rho(x^2+\varepsilon^2)}$ on $z = 0$

[B1] Finally, no flow at $\infty \Rightarrow \phi \to 0$ as $x^2 + z^2 \to \infty$. $\square$

$\boxed{7}$

(b) Let $\hat{\phi}(k, z, t) = \int_{-\infty}^{\infty} \phi(x, z, t) e^{-ikx} \, dx$, $\hat{\eta}(k, t) = \int_{-\infty}^{\infty} \eta(x, t) e^{-ikx} \, dx$

Then IBVP in part (a) implies, by hint,

$$\boxed{\begin{array}{ll} ① & -k^2\hat{\phi} + \hat{\phi}_{zz} = 0 \text{ in } z < 0; \\ ② & \hat{\phi}_z = \hat{\eta}_t, \quad \hat{\phi}_t + g\hat{\eta} = -\frac{p_a}{\rho} e^{-\varepsilon|k|} \text{ on } z = 0; \\ ③ & \hat{\phi} \to 0 \text{ as } z \to -\infty; \\ ④ & \hat{\eta}(k, 0) = \hat{\eta}_t(k, 0) = 0. \end{array}}$$

[S|B4]

The general solution of ① is

$$\boxed{\hat{\phi}(k,z,t) = A(k,t)\, e^{|k|z} + B(k,t)\, e^{-|k|z},}$$

where A, B are arbitrary.

③ $\Rightarrow$ $B = 0$

② $\Rightarrow$ $|k|A = \hat{\eta}_t, \quad A_t + g\hat{\eta} = -\frac{p_a}{\rho} e^{-\varepsilon|k|}$

$$\Rightarrow \hat{\eta}_{tt} + g|k|\hat{\eta} = -\frac{p_a}{\rho}|k| e^{-\varepsilon|k|}$$

$$\Rightarrow \hat{\eta} = C(k)\cos(\omega(k)t) + D(k)\sin(\omega(k)t) - \frac{p_a}{\rho g} e^{-\varepsilon|k|}$$

where C, D are arbitrary and $\omega(k) = \sqrt{g|k|}$

④ $\Rightarrow$ $0 = C(k) - \frac{p_a}{\rho g} e^{-\varepsilon|k|}, \quad 0 = D(k)$

[5/N4] $$\Rightarrow \boxed{\hat{\eta}(k,t) = \frac{p_a}{\rho g} e^{-\varepsilon|k|}\left(\cos(\omega(k)t) - 1\right)}$$

Inverting then gives

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(k,t)\, e^{ikx}\, dk$$

$$\boxed{= \frac{a}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|k|}\left(\cos(\omega(k)t) - 1\right) e^{ikx}\, dk}$$

[5] where $a = p_a/\rho g$. □

$\boxed{9}$

(c) By the hint in part (b), $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|k|} e^{ikx}\, dk = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}$, so

$$\eta(x,t) = \frac{-\varepsilon a}{\pi(x^2+\varepsilon^2)} + \frac{a}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|k|}\frac{1}{2}\left(e^{-i\omega(k)t} + e^{+i\omega(k)t}\right) e^{ikx}\, dk$$

$$\Rightarrow \eta(Vt,t) = -\frac{\varepsilon a}{\pi(V^2t^2+\varepsilon^2)} + \int_{-\infty}^{\infty} f(k)\left(e^{i(kV-\omega(k))t} + e^{i(kV+\omega(k))t}\right)dk$$

[S|N2] where $f(k) = \dfrac{a e^{-\varepsilon|k|}}{4\pi}$.

Thus, $\psi(k) = kV \mp \omega(k)$ in integral in hint.

Note $\omega(k) = g^{1/2}|k|^{1/2} \Rightarrow \dfrac{d\omega}{dk} = \frac{1}{2} g^{1/2}|k|^{-1/2}\,\mathrm{sgn}(k)$

$$\Rightarrow \frac{d^2\omega}{dk^2} = -\frac{1}{4} g^{1/2}|k|^{-3/2} \quad (k \neq 0)$$

Thus $\psi'(k) = V \mp \dfrac{d\omega}{dk} = V \mp \frac{1}{2} g^{1/2}|k|^{-1/2}\,\mathrm{sgn}(k)$, so

$$\psi'(k_*) = 0,\ V > 0 \Rightarrow k_* = \pm \frac{g}{4V^2}.$$

Calculate $f(k_*) = \dfrac{a}{4\pi} e^{-\varepsilon g/4V^2}$

$$\psi(k_*) = \left(\pm\frac{g}{4V^2}\right)V \mp g^{1/2}\left(\frac{g}{4V^2}\right)^{1/2} = \mp \frac{g}{4V}$$

[S|B4] $$\psi''(k_*) = \mp \frac{d^2\omega}{dk^2}(k_*) = \pm \frac{g^{\frac{1}{2}}}{4}\left(\frac{4V^2}{g}\right)^{3/2} = \pm \frac{2V^3}{g}$$

Hence, hint gives, as $t \to \infty$,

$$\eta(Vt,t) + \frac{\varepsilon a}{\pi(V^2t^2+\varepsilon^2)} \sim f(k_*)\left(\exp\left(-\frac{igt}{4V} + \frac{i\pi}{4}\right) + \exp\left(\frac{igt}{4V} - \frac{i\pi}{4}\right)\right) \times \left(\frac{2\pi}{|2V^3/g|\,t}\right)^{1/2}$$

Since $\dfrac{\varepsilon a}{\pi(V^2t^2+\varepsilon^2)} \ll t^{-1/2}$ as $t \to \infty$, we obtain

$$\eta(Vt,t) \sim \frac{A}{t^{1/2}} \cos\left(\frac{gt}{4V} - \frac{\pi}{4}\right)$$

[N|S3] as $t \to \infty$, where $A = \dfrac{Pa}{2\rho(\pi g V^3)^{1/2}} \exp\left(-\dfrac{\varepsilon g}{4V^2}\right)$ □

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2019/B5.4/Q3

(a) (i) Manipulate $\frac{D\rho}{Dt} = -\rho u_x$ ①, $\frac{Du}{Dt} = -\frac{1}{\rho} p_x$ ②, $\rho c_v \frac{DT}{Dt} = -p u_x$ ③, $p = \rho R T$ ④

$$\rho^\gamma \frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{Dp}{Dt} - \frac{\gamma p}{\rho}\frac{D\rho}{Dt}$$

$$= R\rho \frac{DT}{Dt} + RT\frac{D\rho}{Dt} - \gamma RT \frac{D\rho}{Dt} \quad \text{(by ④)}$$

$$= -\frac{Rpu_x}{c_v} + (\gamma-1)\rho R T u_x \quad \text{(by ③ \& ①)}$$

$$= -(\gamma-1)pu_x + (\gamma-1)pu_x \quad (\gamma-1 = R/c_v \text{ \& ④ again})$$

$\Rightarrow \boxed{\frac{D}{Dt}\left(p/\rho^\gamma\right) = 0}$, so p/ρ^γ is conserved following the

[B4] flow $\Rightarrow p/\rho^\gamma = p_0/\rho_0^\gamma$ since uniform initially. □

(ii) $p = k\rho^\gamma \Rightarrow c^2 = \frac{\gamma p}{\rho} = \gamma k \rho^{\gamma-1}$

$$\Rightarrow \rho = Ac^{\frac{2}{\gamma-1}},\ p = \frac{A}{\gamma} c^{\frac{2\gamma}{\gamma-1}}, \text{ where } A = (\gamma k)^{-\frac{1}{\gamma-1}}$$

$$\Rightarrow \rho_t = \frac{2}{\gamma-1}\frac{\rho c_t}{c},\ \rho_x = \frac{2}{\gamma-1}\frac{\rho c_x}{c},\ p_x = \frac{2\gamma}{\gamma-1}\frac{p c_x}{c} = \frac{2}{\gamma-1}\rho c c_x$$

Hence, $\underset{①}{\rho_t + u\rho_x + \rho u_x = 0} \Rightarrow \frac{2}{\gamma-1}(c_t + uc_x) + cu_x = 0$ ⑤

$$\underset{②}{u_t + uu_x + \frac{1}{\rho}p_x = 0} \Rightarrow u_t + uu_x + \frac{2}{\gamma-1}cc_x = 0 \quad ⑥$$

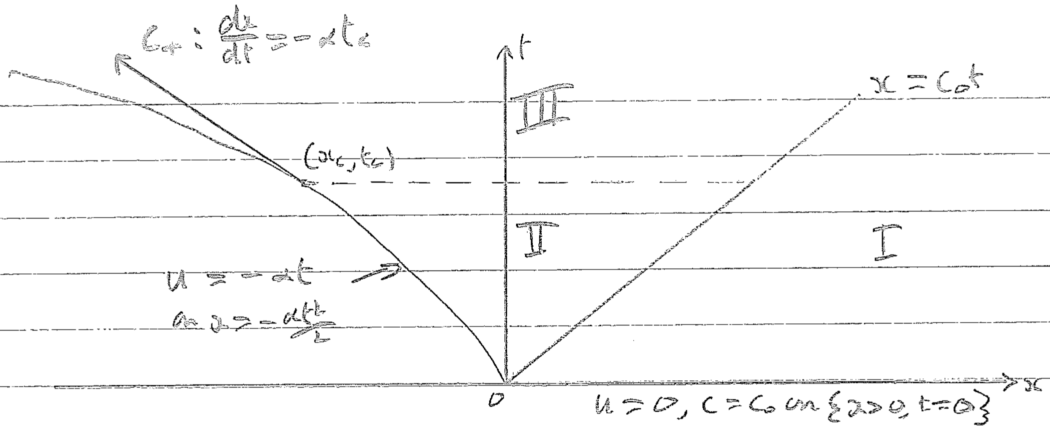
⑤ ± ⑥ $\Rightarrow \boxed{\frac{\partial}{\partial t}\left(u \pm \frac{2c}{\gamma-1}\right) + (u \pm c)\frac{\partial}{\partial x}\left(u \pm \frac{2c}{\gamma-1}\right) = 0}$

Hence, the Riemann invariants $R_\pm = u \pm \frac{2c}{\gamma-1}$

[B6] are constant along $C_\pm$ characteristic curves with $\frac{dx}{dt} = u \pm c$. □

[10]

(b)



(i) Where $C_\pm$ char^{cs} from $\{x>0,\ t=0\}$ intersect,

$$u \pm \frac{2c}{\gamma-1} = 0 \pm \frac{2c_0}{\gamma-1} \Rightarrow \boxed{u=0,\ c=c_0.}$$

Hence such $C_\pm$ char^{cs} are straight lines with $\frac{dx}{dt} = \pm c_0$

[B|S2] and therefore map out the region (I) $x > c_0 t$ for $t>0$.

(ii) On a C_+ char^c originating from $(x,t) = (-\frac{1}{2}\alpha\tau^2, \tau)$ on the piston, $u + \frac{2c}{\gamma-1} = R_+(\tau) = \text{constant}.$

Where this C_+ char^c intersects the family of C_- char^{cs} from $\{x>0,\ t=0\}$, $\boxed{u - \frac{2c}{\gamma-1} = 0 - \frac{2c_0}{\gamma-1}}$,

so that $u = -\alpha\tau$ (by BC on piston) and $c = c_0 + \frac{1}{2}(\gamma-1)(-\alpha\tau)$ on this C_+ char^c, which is therefore a straight line with slope $\frac{dx}{dt} = u+c$, so given by

[S3] $$\boxed{x = -\frac{1}{2}\alpha\tau^2 + \left(c_0 + \frac{1}{2}(\gamma+1)(-\alpha\tau)\right)(t-\tau).}$$

This gives the solution parametrically. Eliminating $\tau = -\frac{u}{\alpha}$

$$\Rightarrow \quad x = -\frac{1}{2}\alpha\left(-\frac{u}{\alpha}\right)^2 + \left(c_0 + \frac{1}{2}(\gamma+1)u\right)\left(t + \frac{u}{\alpha}\right)$$

$$\Rightarrow \quad \gamma u^2 + \left((\gamma+1)\alpha t + 2c_0\right)u + 2\alpha(c_0 t - x) = 0$$

$$\Rightarrow \quad 2\gamma u = -\left((\gamma+1)\alpha t + 2c_0\right) \pm \Big(\underbrace{\left((\gamma+1)\alpha t + 2c_0\right)^2 - 8\gamma\alpha(c_0 t - x)}_{D}\Big)^{\frac{1}{2}}$$

Since $I = 0 \Rightarrow u = 0$, $x > c_0 t$, we must choose the $-$root.

With $t_c = \dfrac{2c_0}{(\gamma-1)\alpha}$, the descriminant is given by

$$
\begin{aligned}
D &= (\gamma+1)^2\alpha^2t^2 + 4(\gamma+1)\alpha c_0 t + 4c_0^2 - 8\gamma\alpha c_0 t + 8\gamma\alpha x + 4\gamma\alpha^2t^2 - 4\gamma\alpha^2t^2 \\
&= (\gamma-1)^2\alpha^2t^2 - 4(\gamma-1)\alpha c_0 t + 4c_0^2 + 4\gamma\alpha(2x + \alpha t^2) \\
&= (\gamma-1)^2\alpha^2(t - t_c)^2 + 4\gamma\alpha(2x + \alpha t^2) \geq 0 \text{ for } x \geq -\frac{\alpha t^2}{2}
\end{aligned}
$$

Moreover, $u = \dfrac{2(c - c_0)}{\gamma - 1}$, so

$$
2\gamma u = \left((\gamma-1)^2\alpha^2(t-t_c)^2 + 4\gamma\alpha(2x + \alpha t^2)\right)^{\frac{1}{2}} - \left((\gamma+1)\alpha t + 2c_0\right) = \frac{4\gamma(c - c_0)}{\gamma - 1} \quad (*)
$$

gives solution in $-\frac{1}{2}\alpha t^2 < x < c_0 t$ and it is physical

[S/N5] so long as $c \propto \rho^{(\gamma-1)/2}$ is non-negative (in region II).

$(*) \Rightarrow u$ and c minimal on piston at $x = -\frac{1}{2}\alpha t^2$, where for $t < t_c$,

$$
\begin{aligned}
c &= c_0 + \frac{\gamma-1}{4\gamma}\left((\gamma-1)\alpha(t_c - t) - (\gamma+1)\alpha t - 2c_0\right) \\
&= c_0 - \frac{\gamma-1}{4\gamma} \cdot 2\gamma\alpha t = \frac{\gamma-1}{2}\alpha(t_c - t),
\end{aligned}
$$

so that the sol$^{\underline{n}}$ $(*)$ is physical for $t < t_c$, but gas density vanishes on piston at time $t = t_c$. □

[S/N2]

(iii) For $t > t_c$ the piston leaves the gas behind forming a vacuum between the piston and the last C_+ char$^{\underline{c}}$ to leave the piston at $(x, t) = (-\frac{1}{2}\alpha t_c^2, t_c)$. On this char$^{\underline{c}}$, $u + \frac{2c}{\gamma-1} = -\alpha t_c$ since $u = -\alpha t_c$, $c = 0$ on piston at $(-\frac{1}{2}\alpha t_c^2, t_c)$, while $u - \frac{2c}{\gamma-1} = -\frac{2c_0}{\gamma-1}$ from C_- char$^{\underline{cs}}$ from $\{x > 0, t = 0\}$

$\Rightarrow u = -\alpha t_c$, $c = 0$ on C_+ char$^{\underline{c}}$ $x = -\frac{1}{2}\alpha t_c^2 + (0 - \alpha t_c)(t - t_c)$.

The analysis in part (ii) still holds for C_+ char$^{\underline{cs}}$ leaving piston before $t = t_c$, so the sol$^{\underline{n}}$ is again given by $(*)$ but now in region $-\frac{1}{2}\alpha t_c^2 - \alpha t_c(t - t_c) < x < c_0 t$ for $t > t_c$ (region III in diagram). □

[N3]

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