2020/B5.4/Q1

(a)(i) All equations satisfied with $\underline{u} = \underline{0}$, $\rho_0 = \rho_a e^{-\beta z}$, $p = p_0(z)$ provided $\nabla p_0 = -\rho_0 g \underline{k}$ in $z < 0$ with $p_0 = p_a$ on $z = 0$.

[B2] Hence, $p_0(z) = p_a - \int_0^z \rho_0(s) g\, ds = p_a + \frac{\rho_a g}{\beta}\left(e^{-\beta z} - 1\right)$.

(a)(ii) The linearized PDEs are (with $\underline{u}' = u'\underline{i} + v'\underline{j} + w'\underline{k}$)

$$\rho'_t + w'\frac{d\rho_0}{dz} \underset{①}{=} 0, \quad \nabla\cdot\underline{u}' \underset{②}{=} 0, \quad \rho_0 \underline{u}'_t \underset{③}{=} -\nabla p' - \rho' g \underline{k}$$

Eliminate u' & v':

$$p'_{xx} + p'_{yy} = -\rho_0 u'_{tx} - \rho_0 v'_{ty} \quad (\text{by } ③)$$
$$= \rho_0 w'_{zt} \quad (\text{by } ②)$$

Since $\frac{d\rho_0}{dz} = -\beta\rho_0$, we are left with

$$\rho'_t \underset{④}{=} \beta\rho_0 w', \quad p'_{xx} + p'_{yy} \underset{⑤}{=} \rho_0 w'_{zt}, \quad \rho_0 w'_t \underset{⑥}{=} -p'_z - \rho' g$$

Now eliminate ρ' and p':

$$(\rho_0 w'_{zt})_{zt} = (\partial_x^2 + \partial_y^2)\, p'_{zt} \quad (\text{by } ⑤)$$
$$= (\partial_x^2 + \partial_y^2)(-\rho_0 w'_{tt} - \rho'_t g) \quad (\text{by } ⑥)$$

$$\Rightarrow \rho_0 w'_{zztt} + \frac{d\rho_0}{dz} w'_{ztt} = (\partial_x^2 + \partial_y^2)(-\rho_0 w'_{tt} - \beta g \rho_0 w') \quad (\text{by } ④)$$

[B5]

$$\Rightarrow (w'_{xx} + w'_{yy} + w'_{zz})_{tt} \underset{(*)}{=} -\beta g \left(w'_{xx} + w'_{yy} - \frac{1}{g} w'_{ztt}\right)$$

(as $\frac{d\rho_0}{dz} = -\beta\rho_0$)

(b) $w' = e^{-i\lambda t} f(x) \sin\left(\frac{\pi z}{h}\right) \Rightarrow (-i\lambda)^2\left(f'' - \frac{\pi^2}{h^2} f\right) = -\beta g f''$

or $f'' + \dfrac{\pi^2}{h^2\left(\frac{\beta g}{\lambda^2} - 1\right)} f = 0$ for $x > 0$ and $\lambda^2 \neq \beta g$.

(i) $\Omega^2 > \beta g \Rightarrow f'' - \nu^2 f = 0, \quad \nu = \dfrac{\pi}{h\left(1-\frac{\beta g}{\Omega^2}\right)^{1/2}} > 0$

$$\Rightarrow f = Ae^{\nu x} + Be^{-\nu x} \quad (A, B \in \mathbb{C} \text{ arb.})$$

BC on $x=0$ & w' bdd at ∞ $\Rightarrow f(0)=a,\ |f(\infty)|<\infty \Rightarrow A=0,\ B=a$

+2 Hence, $w' = a e^{-i\Omega t - \nu x}\sin\left(\frac{\pi z}{h}\right)$, ν as above.

(ii) $\Omega^2 < \beta g \Rightarrow f'' + \mu^2 f = 0, \quad \mu = \dfrac{\pi}{h\left(\frac{\beta g}{\Omega^2}-1\right)^{1/2}} > 0$

$$\Rightarrow f = Ae^{i\mu x} + Be^{-i\mu x} \quad (A, B \in \mathbb{C} \text{ arb.})$$

Hence, $w' = \left(A e^{i(\mu x - \Omega t)} + Be^{-i(\mu x + \Omega t)}\right)\sin\left(\frac{\pi z}{h}\right)$

+2 BC on $x=0$ & radiation condition (no inward waves) $\Rightarrow f(0)=a,\ B=0$

[S6] Hence $w' = a e^{i(\mu x - \Omega t)}\sin\left(\frac{\pi z}{h}\right)$, μ as above

(c)(i) BCs $\Rightarrow w' = \eta_t + u'\eta_x + v'\eta_y,\ p_0(\eta) - p_a + p' = -\gamma\kappa$ on $z=\eta$.

(+1) For small η, Taylor's Theorem gives

$$w'|_{z=\eta} = w'|_{z=0} + \eta w'_z|_{z=0} + \text{h.o.t.}$$

$$p_0(\eta) = p_0(0) + \eta \frac{dp_0}{dz}(0) + \text{h.o.t.}$$

and similarly for u', v' and p'. Hence, the linearized BCs are

(+1, +1) $$w' \underset{(7)}{=} \eta_t, \quad p' \underset{(8)}{=} \rho_a g\eta - \gamma(\eta_{xx}+\eta_{yy}) \text{ on } z=0$$

where we used the fact $\kappa = \nabla^2\eta$ after linearizing for small η

+3 and $p_0(0) = p_a$, $\frac{dp_0}{dz}(0) = -\rho_0(0)g = -\rho_a g$.

Eliminate η e p' :

(+1) $$\rho_a w'_{ztt} = (\partial x^2 + \partial y^2)\, p'_t \qquad \text{(by ⑤ on } z=0\text{)}$$

(+1) $$= \rho_a g(\partial x^2 + \partial y^2)\big(\eta_t - \Gamma(\partial x^2+\partial y^2)\eta_t\big) \qquad \left(\text{by ⑧, } \Gamma = \frac{\gamma}{\rho_a g}\right)$$

(+1) $$= \rho_a g(\partial x^2 + \partial y^2)\big(w' - \Gamma(\partial x^2+\partial y^2)w'\big) \qquad \text{(by ⑦)}$$

+3

[S|NG] giving $$\rho_a w'_{ztt} = \rho_a g(\partial x^2+\partial y^2)w' - \gamma(\partial x^2+\partial y^2)^2 w' \quad \text{on } z=0 \tag{**}$$

(c)(ii) Substituting $w' = A\exp\{ik\cos\alpha\, x + ik\sin\alpha\, y - i\omega t + \lambda z\}$,

$$(*) \Rightarrow (-i\omega)^2\big((ik\cos\alpha)^2 + (ik\sin\alpha)^2 + \lambda^2\big) = -\beta g\left((ik\cos\alpha)^2 + (ik\sin\alpha)^2 - \frac{\lambda(-i\omega)^2}{g}\right)$$

$$(**) \Rightarrow \frac{1}{g}\lambda(-i\omega)^2 = (ik\cos\alpha)^2 + (ik\sin\alpha)^2 - \Gamma\big((ik\cos\alpha)^2 + (ik\sin\alpha)^2\big)^2$$

+2 Hence, $$\omega^2(k^2 - \lambda^2 + \beta\lambda) = \beta g k^2, \quad \lambda\omega^2 = g k^2(1+\Gamma k^2) \tag{+}$$

Eliminate ω^2: $$(1+\Gamma k^2)(k^2 - \lambda^2 + \beta\lambda) = \beta\lambda$$

+2 $$\Rightarrow \lambda^2 - \frac{\beta\Gamma k^2}{1+\Gamma k^2}\lambda - k^2 = 0, \quad 2\lambda = \frac{\beta\Gamma k^2 \pm \big((\beta\Gamma k^2)^2 + 4(1+\Gamma k^2)^2 k^2\big)^{1/2}}{1+\Gamma k^2} \tag{++}$$

But only the +ve root is admissible for $\mathrm{Re}(\lambda) \geq 0$, so that (+) and (++) give the dispersion relation

$$\left\{\beta\Gamma|k| + \big(4(1+\Gamma k^2)^2 + \beta^2\Gamma^2 k^2\big)^{\frac{1}{2}}\right\}\omega^2 = 2g|k|(1+\Gamma k^2)^2,$$

[NG] where we divided through by $|k| = \sqrt{k^2}$.

+2 Thus, $$\omega^2 = \frac{2g|k|(1+\Gamma k^2)^2}{\beta\Gamma|k| + \big(4(1+\Gamma k^2)^2 + \beta^2\Gamma^2 k^2\big)^{1/2}}$$

[N|SG] where $\Gamma = \gamma/\rho_a g$.

2020/B5.4/Q2

+1 (a)(i) No normal flow through rigid walls $\Rightarrow \phi_x = 0$ on $x = 0, a$.

Since $p = p_0 - \rho_0 \phi_t$, the condition of constant
+2 [B2] pressure p_0 on the free ends $\Rightarrow \phi_t = 0$ on $y = 0, b$.

(a)(ii) Seek nontrivial separable solution $\phi = e^{-i\omega t} F(x) G(y)$.

+1 Wave equation $\Rightarrow$ $$-\frac{\omega^2}{c_0^2} - \frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} \quad (FG \neq 0)$$

+1 +2 LHS ind. y & RHS ind. x $\Rightarrow$ LHS = RHS ind. x & y and hence constant.

Hence, $\frac{F''}{F} = -\lambda^2$, $\frac{G''}{G} = -\mu^2$, $\omega^2 = c_0^2(\lambda^2 + \mu^2)$,

where $\lambda, \mu \in \mathbb{R}$ and the signs of the constants have been chosen to give the oscillatory solutions
+1 +2 required for nontrivial solutions for which the
+1 [B/S4] BCs in (a)(i) require $F'(0) = F'(a) = 0$, $G(0) = G(b) = 0$.

For $\lambda = 0$, $F =$ constant, but for $\lambda \neq 0$, $F = A\cos\lambda x + B\sin\lambda x$ ($A, B \in \mathbb{C}$ arb.), so BCs $\Rightarrow B = 0$ and $\lambda a = m\pi$ with
+1 +2 $m \in \mathbb{Z}\setminus\{0\}$. Combo $\Rightarrow F \propto \cos\left(\frac{m\pi x}{a}\right)$, $\lambda = \frac{m\pi}{a}$, $m \in \mathbb{Z}$.

For $\mu = 0$, $G = 0$, but for $\mu \neq 0$, $G = C\cos\mu y + D\sin\mu y$ ($C, D \in \mathbb{R}$ arb.), so BCs $\Rightarrow C = 0$, $\mu b = n\pi$ with
+1 +1 $n \in \mathbb{Z}\setminus\{0\}$. Hence, $G \propto \sin\left(\frac{n\pi y}{b}\right)$, $\mu = \frac{n\pi}{b}$, $n \in \mathbb{Z}\setminus\{0\}$.

Combo $\Rightarrow$ normal modes are $\phi = Ee^{-i\omega t}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$ ($E \in \mathbb{C}$ arb.) with natural frequencies ω s.t.
+1 +1 $$\omega^2 = c_0^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \text{for } m \in \mathbb{Z},\ n \in \mathbb{Z}\setminus\{0\}.$$
[B/S4]

(b)(i) Fourier transform in x : $\hat{\eta}(k,t) = \int_{-\infty}^{\infty} \eta(x,t) e^{-ikx} dx.$

PDE $\Rightarrow$ $\hat{\eta}_{tt} = \beta^2 (ik)^6 \hat{\eta} = -\beta^2 k^6 \hat{\eta}$ for $t>0$.

+3 ICs $\underset{\text{(hint)}}{\Rightarrow}$ $\hat{\eta} = \alpha e^{-\varepsilon^2 k^2/4}$, $\hat{\eta}_t = 0$ at $t=0$.

Hence, $\hat{\eta}(k,t) = A(k)\cos(\omega(k)t) + B(k)\sin(\omega(k)t)$, where $A(k)$, $B(k)$ are arbitrary and $\omega^2 = \beta^2 k^6$, so that $\omega(k) = \beta |k|^3$ wlog.

+2 ICs $\Rightarrow$ $A(k) = \alpha e^{-\varepsilon^2 k^2/4}$, $B(k) = 0$.

Inverse Fourier transform $\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(k,t) e^{ikx} dk$

Hence, $\eta(x,t) = \int_{-\infty}^{\infty} F(k) \cos(\omega(k)t) e^{ikx} dk,$

+1

[S6] where $F(k) = \frac{\alpha}{2\pi} e^{-\varepsilon^2 k^2/4}$ and $\omega(k) = \beta |k|^3$.

(b)(ii) Let $\frac{x}{t} = V = O(1)$ as $x, t \to +\infty$ and use $\cos(\omega t) = \frac{e^{-i\omega t} + e^{i\omega t}}{2}$

$\Rightarrow \eta(Vt,t) = \int_{-\infty}^{\infty} \frac{1}{2} F(k) \left(e^{-i\omega t} + e^{+i\omega t} \right) e^{ikVt} dk$

$\Rightarrow \eta(Vt,t) = \frac{1}{2}\left(I_+(t) + I_-(t)\right)$, where

$$I_\pm(t) = \int_{-\infty}^{\infty} F(k) e^{i\psi_\pm(k)t} dk,$$

+2 $$\psi_\pm(t) = kV \mp \omega(k).$$

[B2] Can now apply the method of stationary phase to $I_\pm(t)$.

The main contribution to $I_\pm(t)$ as $t \to +\infty$ comes from wavenumbers $k = k_*^\pm$, where the phase $\psi_\pm(k)$ is stationary, i.e. $\psi_\pm'(k_*^\pm) = 0$.

Calculate: $\omega = \beta|k|^3 \Rightarrow \omega'(k) = 3\beta k|k|,\ \omega''(k) = 6\beta|k|$.

Hence $\psi_\pm'(k_*^\pm) = 0 \Rightarrow V \mp \omega'(k_*^\pm) = 0 \Rightarrow 3\beta k_*^\pm|k_*^\pm| = \pm V$, giving the critical wavenumbers $k_*^\pm = \pm\left(\frac{V}{3\beta}\right)^{1/2}$

These are each simple zeros because $\psi_\pm''(k_*^\pm) = \mp 6\beta|k_*^\pm| \neq 0$ and ψ_I is twice ctsly diff, while $F(k)$ is ∞ diff and decays exponentially rapidly as $k \to \pm\infty$, so hint applies giving

$$I_\pm(t) \sim F(k_*^\pm)\exp\left\{i\left(\psi_\pm(k_*^\pm)t \mp \frac{\pi}{4}\right)\right\}\left(\frac{2\pi}{|\psi_\pm''(k_*^\pm)|t}\right)^{1/2}$$

as $t \to \infty$ because $\operatorname{sgn}(\psi_\pm''(k_*^\pm)) = \mp 1$

Calculate: $F(k_*^\pm) = \frac{\alpha}{2\pi}e^{-\varepsilon^2V^2/12\beta}$

$$\psi_\pm(k_*^\pm) = \pm\frac{V^{3/2}}{(3\beta)^{1/2}} \mp \beta\left|\frac{V^{1/2}}{(3\beta)^{1/2}}\right|^3 = \pm\frac{2V^{3/2}}{3(3\beta)^{1/2}}$$

$$|\psi_\pm''(k_*^\pm)| = 6\beta\left(\frac{V}{3\beta}\right)^{1/2} = 2(3\beta V)^{1/2}$$

Hence, $I_\pm(t) \sim \frac{\alpha}{2\pi}e^{-\varepsilon^2V^2/12\beta}\exp\left\{\pm i\left(\frac{2V^{3/2}t}{3(3\beta)^{1/2}} - \frac{\pi}{4}\right)\right\}\left(\frac{2\pi}{2(3\beta V)^{1/2}t}\right)^{\frac{1}{2}}$

as $t \to \infty$, giving

$$n(Vt,t) \sim \frac{\alpha e^{-\varepsilon^2V^2/12\beta}}{(4\pi(3\beta V)^{1/2}t)^{1/2}}\cos\left(\frac{2V^{3/2}t}{3(3\beta)^{1/2}} - \frac{\pi}{4}\right)$$

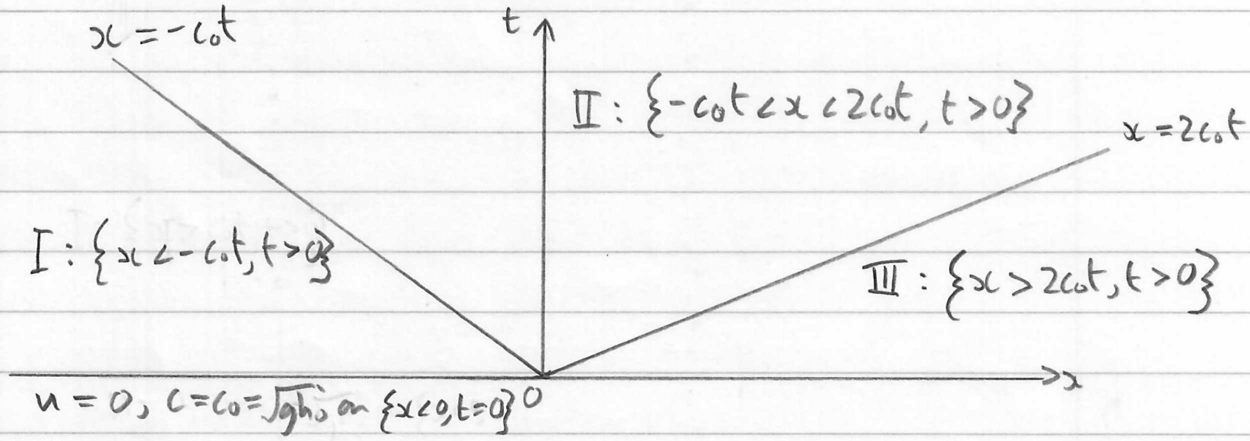
Finally, $e^{-\varepsilon^2V^2/12\beta} \sim 1$ for ε sufficiently small and replace V with x/t elsewhere to obtain

$$n(x,t) \sim \frac{A}{(xt)^{1/4}}\cos\left(\frac{2x^{3/2}}{3(3\beta t)^{1/2}} - \frac{\pi}{4}\right)$$

[N|S7] as $x, t \to +\infty$ with $\frac{x}{t} = O(1)$, where $A = \frac{\alpha}{(4\pi)^{1/2}(3\beta)^{1/4}}$.

B5.4/2020/Q3

(a)(i)



Region I: Where $C_\pm$ char^{cs} from $\{x < 0, t = 0\}$ intersect, $u \pm 2c = \pm 2c_0 \Rightarrow u = 0,\ c = c_0$. Hence such $C_\pm$ char^{cs} are straight lines with $dx/dt = \pm c_0$ and therefore map

[B2] out $x < -c_0 t$, $t > 0$, i.e. $u = 0, c = c_0$ in region I.

Region II: Where a C_- char^c intersects the family of C_+ char^{cs} from $\{x < 0, t = 0\}$, we have $u - 2c =$ const. and $u + 2c = 2c_0$ on the C_- char^c; hence u and c are constant on it and it is therefore straight. To avoid it crossing other C_- char^{cs} in regions I and II, it must originate from the origin and therefore have slope $x/t = u - c$. Solving $u + 2c = 2c_0$, $u - c = x/t$ gives $u = \frac{2}{3}(c_0 + x/t)$ and $c = \frac{1}{3}(2c_0 - x/t)$ in region II, which therefore maps out $-c_0 t < x < 2c_0 t$, $t > 0$ because we require $c, h \geq 0$, i.e. expansion fan terminates on $x = 2c_0 t$ where $c = h = 0$.

[B|S5] Region III: u and c undefined as there is no water.

(a)(ii) The fluid element at $x = -a < 0$ at $t = 0$ moves with the flow according to the ODE

$$\frac{dx}{dt} = u(x,t) = \begin{cases} 0 & \text{for } x < -c_0 t,\ t > 0, \\ \frac{2}{3}\left(c_0 + \frac{x}{t}\right) & \text{for } -c_0 t < x < 2c_0 t,\ t > 0. \end{cases}$$

Hence the element does not move with $x = -a$ until $t = a/c_0$. After this it moves according to

$$\frac{dx}{dt} - \frac{2}{3t}x = \frac{2}{3}c_0 \Rightarrow \frac{d}{dt}\left(\frac{x}{t^{2/3}}\right) = \frac{2c_0}{3t^{2/3}}$$

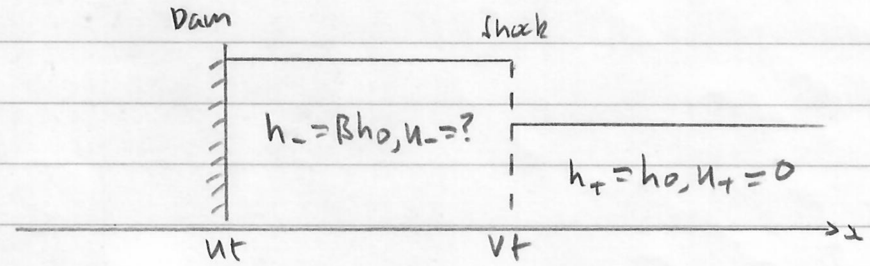
with $x = -a$ at $t = a/c_0$; thus $\frac{x}{t^{2/3}} = 2c_0 t^{1/3} + A$

where $-a\left(\frac{c_0}{a}\right)^{2/3} = 2c_0\left(\frac{a}{c_0}\right)^{1/3} + A$, giving $A = -3a^{1/3}c_0^{2/3}$.

The element therefore reaches the origin at time $t = t_c$
[N5] where $0 = 2c_0 t_c^{1/3} + A \Rightarrow t_c^{1/3} = -A/2c_0 \Rightarrow t_c = \frac{27a}{8c_0}$.

~~(b)(ii) The BC on the dam says that the mass flux of water lost through the dam, i.e. $-\rho h(u-U)|_{x=Ut}$ where ρ is density, is proportional to the force exerted by the water on the dam, i.e. $\int_0^h p - p_{atm}\,dz = \int_0^h \rho g z\,dz \propto h^2$.~~

(b)(i)



Rankine-Hugoniot conditions give

$$\beta h_0(u_- - V) = -h_0 V, \quad \beta h_0(u_- - V)^2 + \tfrac{1}{2}g\beta^2 h_0^2 = h_0 V^2 + \tfrac{1}{2}g h_0^2$$

$$\Rightarrow u_- - V = -\frac{V}{\beta}, \quad \beta\left(-\frac{V}{\beta}\right)^2 + \tfrac{1}{2}g\beta^2 h_0 = V^2 + \tfrac{1}{2}g h_0$$

$$\Rightarrow u_- = \left(1 - \tfrac{1}{\beta}\right)V, \quad V^2\left(1 - \tfrac{1}{\beta}\right) = \tfrac{1}{2}c_0^2(\beta^2 - 1)$$

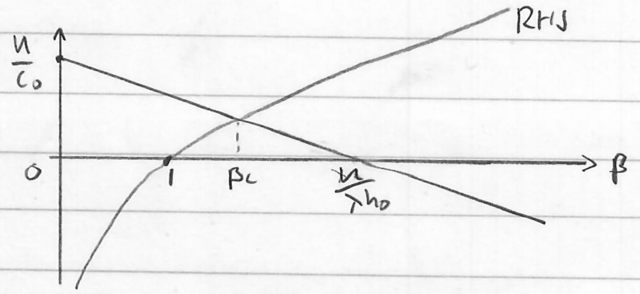
Since we're assuming there is a shock $\beta \neq 1$ and $V > U > 0$,
[B/S3] so taking +ve root gives $V = \frac{c_0}{2^{1/2}}\beta^{1/2}(\beta+1)^{1/2}$

The BC on the dam implies $u_- - U = -\lambda h_-$, giving

$$U - \lambda\beta h_0 = u_- = \frac{\beta-1}{\beta}V = c_0\frac{(\beta-1)(\beta+1)^{1/2}}{(2\beta)^{1/2}}$$

i.e. $$\frac{(\beta-1)(\beta+1)^{1/2}}{(2\beta)^{1/2}} = \frac{U}{c_0} - \frac{\lambda h_0}{c_0}\beta. \qquad (+)$$

Since the LHS ↗ with $\beta>0$ and RHS ↘ with $\beta>0$ as illustrated, $\exists!$ root $\beta=\beta_c$ to $(+)$ as illustrated.



[S/N3]

(b)(ii) Rate at which energy flows out of a stationary shock is

$$Q = \Big[\underbrace{\int_0^h \left(\tfrac{1}{2}\rho u^2 + \rho g z\right)u\,dz}_{\text{Rate at which KE \& PE } \nearrow} + \underbrace{\int_0^h (p-p_{atm})u\,dz}_{\text{Rate at which work is done by pressure.}}\Big]_-^+$$

Since $p - p_{atm} = \rho g z$ in shallow water theory and $[hu]_-^+=0$

$$Q = \left[\tfrac{1}{2}\rho h u^3 + \rho g h^2 u\right]_-^+ = \rho \underbrace{h_\pm u_\pm}_{\text{Conserved}}\left[\tfrac{1}{2}u^2 + gh\right]_-^+$$

Hence for our moving shock, replacing u with $u-V$,

$$Q = \rho h_-(u_- - V)\left[\tfrac{1}{2}(u-V)^2 + gh\right]_-^+ = \rho h_0(-V)\left(\tfrac{1}{2}V^2 + gh_0 - \tfrac{1}{2}\left(-\tfrac{V}{\beta}\right)^2 - g\beta h_0\right)$$

[S/N5] $$\Rightarrow Q = -\rho h_0 V\left(\tfrac{1}{2}\tfrac{gh_0}{2}\beta(\beta+1)\left(1-\tfrac{1}{\beta^2}\right) + gh_0(1-\beta)\right) = \rho g h_0^2 V \underbrace{\frac{(1-\beta)^3}{4\beta}}_{q}$$

The shock can only be physical if energy lost as fluid crosses the shock (i.e. energy cannot be created), which requires $q<0$. But $q<0$

[N2] $\Rightarrow \beta_c > 1 \Rightarrow \frac{U}{\lambda h_0} > \beta_c > 1$ (from diagram) $\Rightarrow U > \lambda h_0$.