Plan of the lecture

1. Introduction and history
2. Review: classical and quantum mechanics
3. Classical field theory
4. Canonical quantization
5. Path integral formulation of QFT
6. Feynman diagrams and Feynman rules
7. Divergencies and regularization
8. Renormalization and renormalization group
9. Scattering processes

Literature:

- M. Srednicki – „Quantum field theory"
- M. Peskin, D. Schroeder – „An Introduction to QFT"
- L. Ryder – „Quantum field theory"
- (*) A. Zee – „Quantum field theory in a nutshell"
- (*) R. Ticciati – „QFT for Mathematicians"

Online resources:

- D. Tong – Lectures on Quantum Field Theory, Cambridge
- N. Beisert – Quantum Field Theory 1, ETH Zurich

What is QFT?

- Quantization of a classical field (e.g. electromagnetic field)
- Field: quantity defined at every point in space and time $(\vec{x}, t)$
- In QFT fields are primary object and particles are derived concepts appearing after quantization as excitations of quantum fields

Pre-QFT world

Classical mechanics

- Newton's laws
- particles travel along classical trajectories

$\hbar \to 0$

$c \to \infty$

Quantum mechanics

- A state of the system is represented by a vector in Hilbert space
- Observables are Hermitian operators
- A measurement of an observable yields one of its eigenvalues as the result

Relativistic mechanics

- Laws of physics are the same in all inertial frames
- Speed of light constant for all observers
- Space and time on the same footing

Classical FT

- Describe propagation of waves along solutions to field equations
- e.g. Maxwell's theory of electromagnetism

$$\boxed{\text{Quantum field theory}}$$

History

- Maxwell's equations
  - a classical theory of electromagnetism
- Planck hypothesis:
  - the photon as quantum of radiation
- Quantum mechanics
  - applied to the Maxwell's field predicts photon (problem with relativistic quantum mechanics: negative energy states)
- Relativity + quantum mechanics = many-particle description
  - particles are the quanta of a quantized classical field theory, in analogy with photon
- Formulation of calculation rules for QED
  - Feynman diagrams
- Formulation of the path integral approach
- Understanding of how to deal with divergencies of Feynman diagrams through renormalization
- QFT has problems: how to apply it to weak and strong interactions?
- Renormalization of non-abelian gauge theories
- Renormalization group (RG)
- Asymptotic freedom
- Standard Model
- String theory and quantum gravity
- Conformal field theories
- All QFTs are only effective over some range of lengths and scales
- Holography
- Strong coupling results for gauge theories

1900 1920 1930 1940 1950 1960 1970 1980 1990 2000

Necessity of the field viewpoint

Problems with particle description:

- Klein-Gordon eq. or Dirac eq. $\longrightarrow$ negative energy states
- Particle-antiparticle creation : single particle description not consistent with perturbation theory $\rightarrow$ even with a small amount of energy $(\Delta E \cdot \Delta t \sim \hbar)$ pairs of particles exist for a very small amount of time

- Causality:

  Consider a free particle propagating from x_0 to x

$$\mathcal{U}(t) = \langle x | e^{-iHt} | x_0 \rangle$$

- in non-relativistic case $H = \frac{p^2}{2m}$. Then

$$\mathcal{U}(t) = \langle x | e^{-i\frac{p^2}{m}t} | x_0 \rangle = \int \frac{d^3p}{(2\pi)^3} \langle x | e^{-i\frac{p^2}{m}t} | p \rangle \langle p | x_0 \rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3p \, e^{-i\frac{p^2}{m}t} e^{ip(x-x_0)}$$

$$= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{im(x-x_0)^2/2t}$$

It is non-zero for any x and t $\longrightarrow$ particles can propagate between any two points in an arbitrarily small time $\longrightarrow$ violation of causality

- relativistic case : $H = \sqrt{p^2+m^2}$ $\longrightarrow$ the same problems
- Solution: Propagation of a particle across a space-like interval is indistinguishable from the propagation of an anti-particle in the opposite direction
  $\rightarrow$ amplitudes for particles and antiparticles cancel

Features of QFTs

- Locality: in classical physics, primary reason to introduce a field is to construct laws of Nature which are local:
  Coulomb and Newton laws involve „action at a distance"
  - forces change immediately → wrong!

  Interactions mediated in local fashion by a field

- Particle number is not conserved: first predicted by Dirac who understood how relativity implies the necessity of antiparticles.
  Presence of particles and antiparticles at short distances tells us that any attempt to write down a relativistic version of the 1-particle Schrödinger eq. fails

- Infinities: One reason why QFT was claimed to be mathematically ill-defined. Origin of infinities:
  - infinitely extended fields: source of IR infinities (low energy)
  - infinite spatial resolution: source of UV infinities (high energy)

- Interactions: possible interactions in QFT governed by few basic principles: locality, symmetry, renormalization group flow (= decoupling of short distance phenomena from physics at large scales)
  - given set of fields, often almost unique way to couple them: QFT is very robust framework

Review: classical mechanics

- Two equivalent descriptions of classical mechanics:

Newton's e.o.m.	Action principle
iterate small changes along a particle's path	evaluate all possible paths between 2 points and select out the one which minimize S

- Equivalence breaks down in quantum realm
  - uncertainty principle forces us to introduce probabilities, with classical paths being the most likely
  - deviation from classical path is very small, on the scale determined by Planck's constant
- Newtonian mechanics fails but one can generalize action principle to incorporate quantum probabilities.

Lagrangian formulation of classical mechanics

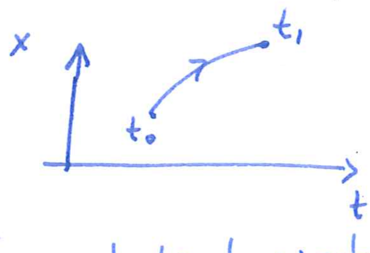
- Lagrangian for a point particle with mass m:

$$L = \frac{1}{2} m \dot{x}^2 - \underbrace{V(x)}_{\text{potential}}$$

$$\dot{x} = \frac{dx}{dt}$$

- Principle of least action:

$$S = \int_{t_0}^{t_1} dt \, L(x, \dot{x})$$



Consider a small variation of particle trajectory

$$x(t) \longrightarrow x(t) + \delta x(t) \qquad \frac{\delta x}{x} \ll 1$$

with endpoints fixed

$$\delta x(t_0) = \delta x(t_1) = 0$$

• Trajectory of particle is the one for which $\delta S = 0$

$$S + \delta S = \int_{t_0}^{t_1} L(x+\delta x, \dot{x} + \delta \dot{x})\, dt$$

$$= \int_{t_0}^{t_1} \left[L(x,\dot{x}) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \dots \right] dt$$

$$= S + \frac{\partial L}{\partial \dot{x}}\delta x \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \underbrace{\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right]}\, dt\, \delta x$$

$$\delta S = 0 \rightarrow \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

Euler-Lagrange eq.

• For a point particle:

$$m\ddot{x}(t) = -V'(x(t))$$ ← Newton's second law

• Lagrangian formalism and Euler-Lagrange equation easily generalize to other systems: in any number of dimensions, multi-particle systems, systems with infinite number of degrees of freedom.

• Lagrangian formalism makes symmetries explicit

Hamiltonian formulation of classical mechanics

• Define the conjugate momentum p:

$$p = \frac{\partial L}{\partial \dot{x}}$$

and Hamiltonian on phase space (x,p):

$$H = p\dot{x} - L(x,\dot{x})$$ ← total energy of system

• Hamilton's equations:

$$\frac{\partial H}{\partial x} = -\dot{p}\,, \quad \frac{\partial H}{\partial p} = \dot{x}$$ ← equivalent to Euler-Lagrange equation but 1st order

Review: quantum mechanics

- Canonical quantization (2 steps):

1. Dynamical variables of a system are replaced by operators

$$\text{position: } x \longrightarrow \hat{x}$$

$$\text{momentum: } p \longrightarrow \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$\text{Hamiltonian: } H \longrightarrow \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = -\frac{\hbar^2 \nabla^2}{2m} + V(\hat{x})$$

2. Impose commutation relations on these operators

$$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

- Physical state of a quantum mechanical system is encoded in state vectors $|\psi\rangle$ which are elements of a Hilbert space $\mathcal{H}$
- Hermitian conjugate $\langle\psi| = (|\psi\rangle)^+$
- Mod square of scalar product between 2 states: probability for system to go from 1 to 2

$$|\langle\psi_1|\psi_2\rangle|^2 = \text{prob for } |\psi_1\rangle \rightarrow |\psi_2\rangle$$

- Physical observables $\hat{O}$ given by expectation values of hermitian operators $\hat{O} = \hat{O}^+$

$$O_{12} = \langle\psi_2|\hat{O}|\psi_1\rangle$$

Hermiticity $\rightarrow$ expectation values are real

- State vector cannot be simultaneous eigenstate of non-commuting operators: Heisenberg uncertainty principle

$$\Delta A\, \Delta B \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|$$

Time evolution of systems

- Schrödinger picture:

$$\begin{cases} \text{state vectors are functions of time: } |\psi(t)\rangle \\ \text{operators do not evolve in time} \end{cases}$$

Time evolution of system described by Schrödinger eq.

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

If at t_0, system is in state $|\psi(t_0)\rangle$ then

$$|\psi(t)\rangle = \underbrace{\exp\left\{-\frac{i}{\hbar}\hat{H}(t-t_0)\right\}}_{\text{time evolution operator}} |\psi(t_0)\rangle$$

- Heisenberg picture:

$$\begin{cases} \text{state vector constant} \\ \text{operators evolve in time} \end{cases}$$

$$\hat{O}_H(t) \equiv e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\, \hat{O}_S\, e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$$

Heisenberg equation:

$$i\hbar \frac{d\hat{O}_H(t)}{dt} = [\hat{O}_H, H]$$