Review - special relativity

- Minkowski space : x^μ , $\mu = 0,1,2,3$
  - $x^\mu = (t, x^1, x^2, x^3) = (t, \vec{x})$
  - $p_\mu = (E, p_1, p_2, p_3) = (E, \vec{p})$
- Signature :

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$$

- Einstein summation convention :

$$p^2 = p_\mu p^\mu \equiv \sum_{\mu=0,1,2,3} p_\mu p^\mu = E^2 - |\vec{p}|^2$$

Review - mathematical methods

- Heaviside step function $\Theta(x)$

$$\Theta(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x > 0 \end{cases}$$

- Dirac delta function $\delta(x)$

$$\delta(x) = \frac{d}{dx} \Theta(x)$$

In n dimensions :

$$\int d^n x \, \delta^{(n)}(x) = 1$$

- Fourier transform

$$f(x) = \int \frac{d^n k}{(2\pi)^n} e^{-ik\cdot x} \tilde{f}(k)$$

$$\tilde{f}(k) = \int d^n x \, e^{ik\cdot x} f(x)$$

- Cauchy theorem and contour integrals

From Quantum Mechanics to Classical Field Theory

- Schrödinger equation is not relativistic:

$$-i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

Time and space treated differently.

- Non-relativistic Hamiltonian for a free particle

$$H = \frac{\vec{p}^{\,2}}{2m}$$

can be enhanced to the relativistic case:

$$H = mc^2\sqrt{1+\frac{\vec{p}^{\,2}}{m^2c^2}} = \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{\vec{p}^{\,2}}{2m}}_{\text{non-relativistic energy}} + \underbrace{\mathcal{O}\left(\tfrac{1}{c}\right)}_{\text{higher-order terms}}$$

This suggests the „relativistic Schrödinger equation"

$$-i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle = \sqrt{m^2c^4 + c^2\vec{p}^{\,2}}\, |\psi(t)\rangle$$

→ difficult to make sense of this equation because of the $\sqrt{}$

→ not Lorentz invariant (time treated differently from space)

- Way out: square the relativistic Schrödinger eq. (in position space)

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\vec{x},t) = \left(m^2c^4 - \hbar^2c^2 \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}\right) \psi(\vec{x},t)$$

$$\rightarrow \left(\partial_\mu \partial^\mu + \frac{m^2c^2}{\hbar^2}\right) \psi(\vec{x},t) = 0 \qquad \text{Klein-Gordon eq.}$$

- manifestly relativistic
- second order in time derivatives

- From now on we use:

$$\boxed{\hbar = c = 1}$$

Review: harmonic oscillator

- Classically:

$$H = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 x^2\right)$$

Using Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

Equation of motion:

$$\ddot{x} + \omega^2 x = 0$$

wave equation

- Quantum mechanically:

$$x \to \hat{x} \quad , \quad p \to \hat{p}$$

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2 \hat{x}^2\right)$$

Instead of solving differential equations, we use an algebraic approach:

- introduce new operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\sqrt{\frac{\hbar}{m\omega}}\,\hat{p}\right)$$ annihilation operator

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - i\sqrt{\frac{\hbar}{m\omega}}\,\hat{p}\right)$$ creation operator

Use commutation relations for $\hat{x}, \hat{p}$ $\longrightarrow$

$$[\hat{a}, \hat{a}^+] = 1 \quad , \quad [\hat{H}, \hat{a}] = -\hbar\omega\hat{a} \quad , \quad [\hat{H}, \hat{a}^+] = \hbar\omega\hat{a}^+$$

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+\right) = \hbar\omega\left(\hat{a}^+ a + \frac{1}{2}\right)$$

Spectrum of harmonic oscillator

- Introduce a ground state which is annihilated by $\hat{a}$

$$\hat{a}|0\rangle = 0$$

- Energy of the ground state:

$$\hat{H}|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$$

- Excited states:

$$|n\rangle = (\hat{a}^{+})^{n}|0\rangle \qquad n = 1, 2, \dots$$

with energies

$$\hat{H}|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$$

Canonical quantization of free scalar field $\phi(\vec{x},t)$

Hamiltonian:

$$H = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Before quantizing this Hamiltonian let us first observe that it can be related to the harmonic oscillator.

In momentum space:

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t)$$

Then, the Klein-Gordon equation become

$$\left[\frac{\partial^2}{\partial t^2} + (|\vec{p}|^2 + m^2)\right]\phi(\vec{p},t) = 0$$

This is the eom for a single harmonic oscillator with frequency

$$\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$$

Quantization procedure:

① Introduce conjugate momentum:

$$\Pi(\vec{x},t) = \frac{\partial}{\partial t}\phi(\vec{x},t)$$

② Treat ϕ and Π as independent variables and promote them to operators. The Hamiltonian:

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

③ Impose canonical commutation relations (in Schrödinger picture)

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

④ Similar to the discussion for harmonic oscillators we can write these fields in terms of creation and annihilation operators

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + a^+_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\Pi(\vec{x}) = (-i)\int \frac{d^3\vec{p}}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} - a^+_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\right)$$

with commutators

$$[a_{\vec{p}}, a^+_{\vec{p}'}] = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}')$$

⑤ Hamiltonian

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\omega_{\vec{p}}\left(a^+_{\vec{p}}a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a^+_{\vec{p}}]\right)$$

proportional to $\delta(0)$
(sum of zero-point energies)
cannot be detected since
we measure only energy
differences

Commutation with Hamiltonian

$$[H, a_{\vec{p}}^{+}] = \omega_{\vec{p}}\, a_{\vec{p}}^{+} \quad , \quad [H, a_{\vec{p}}] = -\omega_{\vec{p}}\, a_{\vec{p}}$$

⑥ Spectrum of Hamiltonian

→ vacuum (ground state)

$$a_{\vec{p}}\,|0\rangle = 0 \qquad \forall \vec{p}$$

with energy $E = 0$ (after we drop the infinite term)

→ excited states

$$|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}}\; a_{\vec{p}}^{+}|0\rangle \quad , \quad H|\vec{p}\rangle = \omega_{\vec{p}}|\vec{p}\rangle$$

$$|\vec{p}_1, \vec{p}_2\rangle = \mathcal{N}\, a_{\vec{p}_1}^{+}\, a_{\vec{p}_2}^{+}|0\rangle , \quad H|\vec{p}_1, \vec{p}_2\rangle = (\omega_{\vec{p}_1} + \omega_{\vec{p}_2})|\vec{p}_1, \vec{p}_2\rangle$$

Excitations are called particles.

We choose the normalization of the one-particle state such that:

$$\langle \vec{p}\,|\,\vec{q}\rangle = 2E_{\vec{p}}\,(2\pi)^3\,\delta^{(3)}(\vec{p} - \vec{q}) \qquad \text{Lorentz invariant}$$

Interpretation of the state corresponding to the field $\phi(\vec{x})$:

$$\phi(\vec{x})|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2E_{\vec{p}}}\, e^{-i\vec{p}\cdot\vec{x}}\,|\vec{p}\rangle$$

superposition of single-particle states with well-defined momentum → The operator $\phi(\vec{x})$ creates a particle at given position $\vec{x}$.

In particular

$$\langle 0|\,\phi(\vec{x})\,|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}}$$

Heisenberg picture: better for studying time-dependent quantities and causality.

$$\phi(x) = \phi(\vec{x}, t) = e^{iHt}\, \phi(\vec{x})\, e^{-iHt}$$

- Heisenberg equation of motion

$$i \frac{\partial}{\partial t} \mathcal{O} = [\mathcal{O}, H]$$

allows to compute the time-dependence of ϕ and π:

$$i \frac{\partial}{\partial t} \phi(x) = i\, \pi(x) \quad , \quad i \frac{\partial}{\partial t} \pi(x) = -i\,(-\nabla^2 + m^2)\, \phi(x)$$

Combining it together we get the Klein-Gordon eq.

$$\frac{\partial^2}{\partial t^2} \phi(x) = (\nabla^2 - m^2)\, \phi(x)$$

- Decomposition of fields in Heisenberg picture:

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}\, e^{ip\cdot x} \right) \Big|_{p^0 = E_{\vec{p}}}$$

$$\pi(x) = \frac{\partial}{\partial t} \phi(x)$$

Causality

In the Heisenberg picture we can study amplitudes for a particle to propagate from y to x

$$\langle 0 | \phi(x)\, \phi(y) | 0 \rangle \equiv D(x-y)$$

Explicitly:

$$D(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

- For time-like separation : $x^0 - y^0 = t$, $\vec{x} - \vec{y} = 0$

$$D(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\underset{t\to\infty}{\sim} e^{-imt}$$

- For space-like separation : $x^0 - y^0 = 0$, $\vec{x} - \vec{y} = \vec{r}$

$$D(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_{\vec{p}}} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{+\infty} dp \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}} = (*)$$