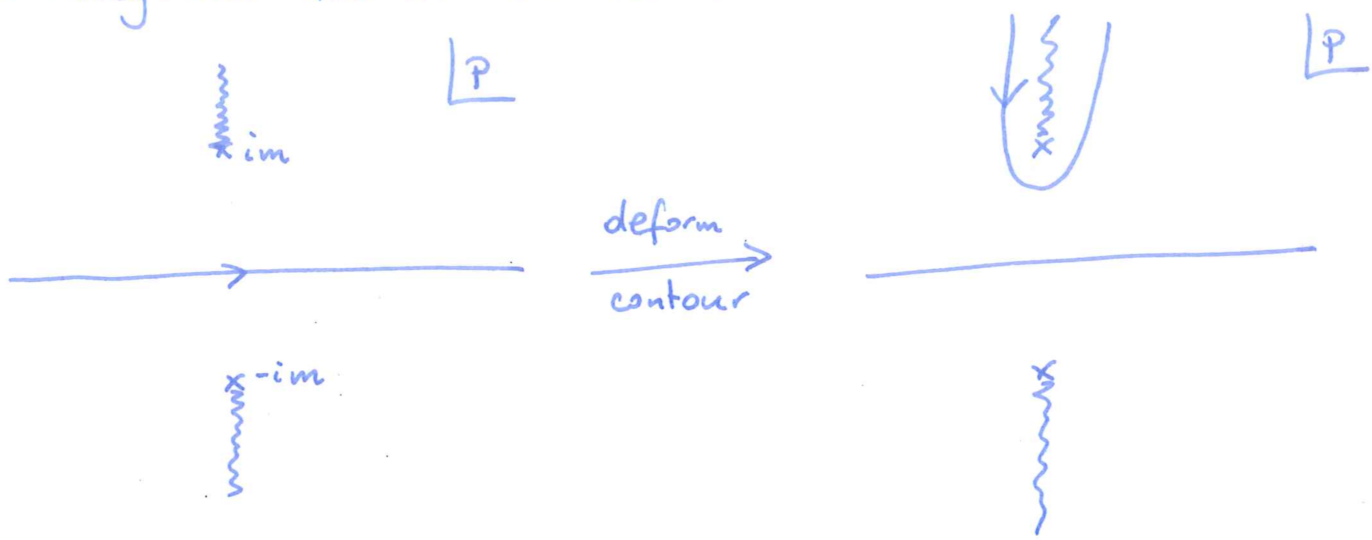
The integrand has branch cuts:



Change variables $\rho = -ip$

$$(*) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \underset{r\to\infty}{\sim} e^{-mr}$$

Exponentially vanishing but not zero!

- Causality is however about something else:

$$\boxed{\text{Can a measurement performed at one point affect a measurement at another space-like separated point}}$$

In order to measure it we study the commutator $[\phi(x),\phi(y)]$: if it is zero then one measurement cannot affect the other

- We already know that for $x^0=y^0 \to [\phi(x),\phi(y)]=0$

In full generality:

$$[\phi(x),\phi(y)] = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{q}}}} \times$$

$$\times\left[a_{\vec{p}}e^{-ipx}+a^+_{\vec{p}}e^{ipx},\ a_{\vec{q}}e^{-iqy}+a^+_{\vec{q}}e^{iqy}\right]$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(e^{-ip(x-y)}-e^{ip(x-y)}\right)$$

$$= D(x-y) - D(y-x)$$

- For $(x-y)^2 < 0$ there exists a Lorentz transformation (proper, orthochronous) taking $x-y$ to $-(x-y)$. Then $D(x-y) - D(y-x) = 0$ and causality is preserved
- For $(x-y)^2 > 0$ there is no such tranformation and $[\phi(x), \phi(y)] \sim e^{-imt} - e^{imt} \neq 0$

Klein-Gordon propagator

Let us study the commutator $[\phi(x), \phi(y)]$ further.

For $x^0 > y^0$

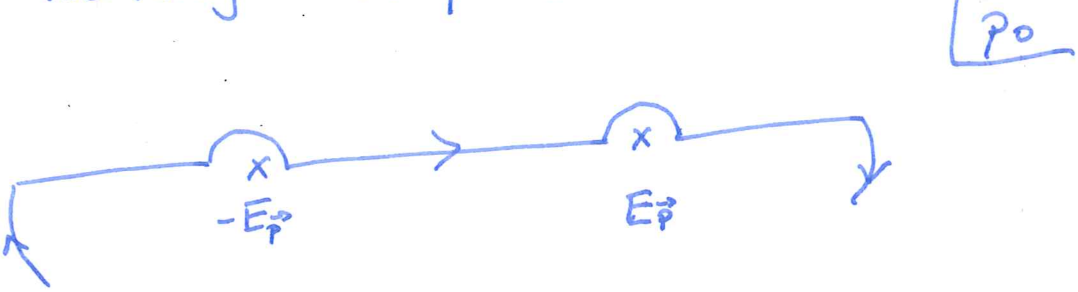
$$[\phi(x), \phi(y)] = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left\{ \frac{1}{2E_{\vec{p}}} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) \Big|_{p_0 = E_{\vec{p}}} \right.$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left\{ \frac{1}{2E_{\vec{p}}} e^{-ip(x-y)} \Big|_{p_0 = E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} e^{-ip(x-y)} \Big|_{p_0 = -E_{\vec{p}}} \right\}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip(x-y)}$$

where the integral in p^0 is over the contour



For $x^0 < y^0$ this quantity is 0. Then

$$D_R(x-y) \equiv \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

Let us understand this quantity better:

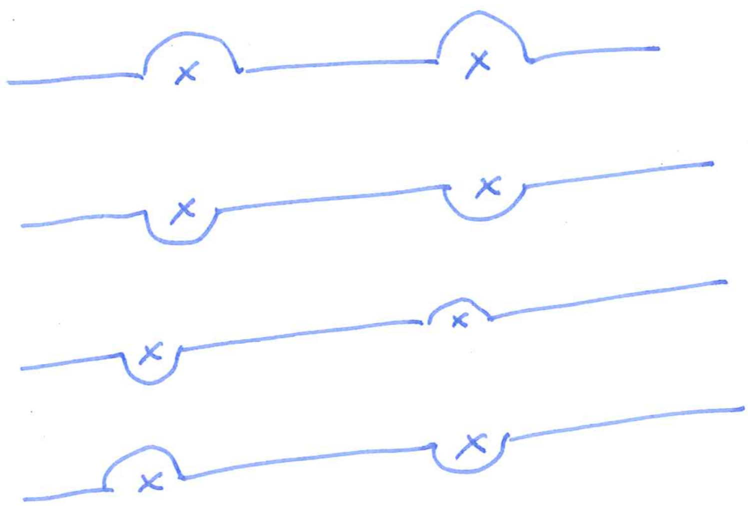
$$
\begin{aligned}
(\partial^2+m^2)D_R(x-y) &= \{\partial^2\theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle \\
&+ 2\,\partial_\mu\theta(x^0-y^0)\,\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle \\
&+ \theta(x^0-y^0)(\partial^2+m^2)\langle 0|[\phi(x),\phi(y)]|0\rangle \\
&= -i\,\delta^{(4)}(x-y)
\end{aligned}
$$

It is a Green's function of the Klein-Gordon eq.
We call it the retarded Green's function.

- General Green's function for Klein-Gordon eq.

$$D(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)}$$

There are four different contours over which we can perform the p_0 integral:



retarded

advanced

Feynman

Feynman propagator

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip(x-y)}$$

$$= \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle$$

where $T\{\}$ means time ordering of operators.

Lagrangian Field Theory and Action Principle

- Fundamental quantity of classical mechanics: action S

In local field theory:

$$S = \int L \, dt = \int \mathcal{L}(\phi, \partial_\mu \phi) \, d^4x$$

Lagrangian density

- Principle of least action:

System evolves from one given configuration to another between times t_1 and t_2 along path in configuration space for which S is an extremum.

$$0 = \delta S = \int d^4x \left\{ \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi \right) \right\}$$

surface integral over the boundary of 4D region of integration

vanish for tempered variations →

→ Euler-Lagrange e.o.m.

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) - \frac{\delta \mathcal{L}}{\delta \phi} = 0$$

- For free scalar particle:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \longrightarrow \text{Klein-Gordon eq.}$$

Noether's theorem

- Symmetries in FT: – restrict classes of models
  – provide stability
  – simplify calculation

In fundamental particle physics all models have the relativistic invariance.

- Symmetry: a transformation of fields

$$\phi \to \phi'$$

such that solutions of the e.o.m. are mapped to other solutions of e.o.m.

Example: complex scalar field

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi$$

e.o.m.: $(\partial^\mu \partial_\mu + m^2)\phi = 0$

$(\partial^\mu \partial_\mu + m^2)\phi^* = 0$

Consider a global transformation:

$$\phi'(x) = e^{i\alpha}\phi(x), \quad \phi^{*\prime}(x) = e^{-i\alpha}\phi^*(x)$$

This transformation maps solutions to solutions:

$$\partial^\mu \partial_\mu \phi' + m^2 \phi' = e^{i\alpha}(\partial_\mu \partial^\mu \phi + m^2 \phi) = 0$$

Moreover:

$$\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi) \text{ and } S[\phi'] = S[\phi]$$

It is a symmetry of the action → more powerful than symmetries of equation of motion.

Noether's theorem

- Every continuous symmetry of the action leads to a conserved current $\longrightarrow$ conserved charge
- Derivation of Noether's theorem:

  - consider an infinitesimal variation of field configuration

$$\phi \to \phi' = \phi + \alpha\, \delta\phi$$

small parameter

  - we assume that this transformation is a symmetry in the strong sense, namely $S'[\phi'] = S'[\phi]$
  - invariance of the action implies that the Lagrangian density is invariant up to a 4-divergence

$$\mathcal{L} \longrightarrow \mathcal{L} + \underbrace{\alpha\, \partial_\mu J_0^\mu}_{\delta\mathcal{L}}$$

  - we can compare it against direct variation of fields:

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\delta\phi$$

$$\overset{\text{e.o.m.}}{=} \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\delta\phi = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi\right)$$

  - Then:

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi\right) = \partial_\mu J_0^\mu$$

$$\rightarrow \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi - J_0^\mu\right) = 0$$

  - Define a current

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi - J_0^\mu$$

It is conserved: $\partial_\mu J^\mu = 0$

Noether's theorem

- Define a charge

$$Q(t) = \int d^3\vec{x}\ J^0(t,\vec{x})$$

It is conserved:

$$\dot{Q}(t) = \int d^3\vec{x}\ \partial_0 J^0(t,\vec{x}) = -\int d^3\vec{x}\ \partial_i J^i = 0$$

- Example: complex scalar field

$$\delta\phi = i\phi \quad , \quad \delta\phi^* = -i\phi^*$$

The Lagrangian is invariant $\delta\mathcal{L} = 0 \Rightarrow J_0^\mu = 0$

The other term:

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\delta\phi^*$$

$$= (\partial^\mu\phi^*)(i\phi) + (\partial^\mu\phi)(-i\phi^*)$$

$$= i(\partial^\mu\phi^*\cdot\phi - \phi^*\partial^\mu\phi)$$

Conserved since:

$$\partial_\mu J^\mu = i(\partial^\mu\partial_\mu\phi^*\cdot\phi - \phi^*\partial^\mu\partial_\mu\phi) \overset{\text{e.o.m}}{=} 0$$

Conserved charge:

$$Q = i\int d^3\vec{x}\,(\dot{\phi}^*\phi - \phi^*\dot{\phi})$$

In quantum theory it becomes:

$$Q = N_{\text{particle}} - N_{\text{antiparticle}}$$

(it is conserved!) particles can only be created or destroyed in pairs with antiparticles