Feynman rules in momentum space

Feynman rules are more natural in the momentum space

$$\Delta(x'-x'') = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip\cdot(x'-x'')}}{p^2+m^2}$$

Define the Fourier transform of Green's functions:

$$\int G^{(N)}(y_1,\dots,y_N)_c\, e^{i(p_1 y_1+\dots+p_N y_N)}\, d^D y_1 \dots d^D y_N$$

$$\equiv \tilde{G}^{(N)}(p_1,\dots,p_N)_c\, (2\pi)^D\, \delta^{(D)}(p_1+\dots+p_N)$$

↑ must be here since $G^{(N)}$ depends only on differences of points positions

- Then for each internal vertex:

$$\int d^D x_j\, e^{i\left(\sum_k p_{kj}\right)x_j} = (2\pi)^D \delta^{(D)}\left(\sum_k p_{kj}\right)$$

→ momentum is conserved at each vertex.

Feynman rules

1. Draw all topologically distinct connected diagrams with N external lines and each internal vertex attached to 4 lines.
2. Assign momenta flowing along each line so that the external lines have momenta $\{p_i\}$ and the momentum is conserved at each vertex
3. To each vertex associate a factor $-\lambda$ $[-i\lambda]$
4. To each line associate a factor $(p^2+m^2)^{-1}$ $\left[\frac{i}{p^2-m^2+i\epsilon}\right]$
5. Integrate over remaining loop momenta $\prod_j \int \frac{d^D p}{(2\pi)^D}$
6. Multiply by the symmetry factor $1/\text{integer}$

Feynman rules for other QFT's

- All Feynman rules are based on building blocks: propagators and vertices
- Example: complex scalar field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) + m^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2$$

- Propagators:

$$\langle \phi(x_1)\, \phi^*(x_2) \rangle_0 = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip(x_1 - x_2)}}{p^2 + m^2} = \Delta(x_1 - x_2)$$

$$\langle \phi(x_1)\, \phi(x_2) \rangle_0 = \langle \phi^*(x_1)\, \phi^*(x_2) \rangle_0 = 0$$

can be traced back to the $U(1)$ symmetry of the action

$$\phi \to e^{i\alpha} \phi \,,\ \phi^* \to e^{-i\alpha} \phi^*$$

In Minkowski space $\langle \phi(x_1)\, \phi^*(x_2) \rangle$ describes the propagation of a particle if $t_1 < t_2$ or of an antiparticle if $t_1 > t_2$.

- Feynman diagrammatics:

$$\langle \phi(x_1)\, \phi^*(x_2) \rangle_0 = x_1 \longrightarrow x_2$$

- There are different symmetry factors for this theory:

 $\longrightarrow S_G = 1$ (instead of $\frac{1}{2}$ for real scalar field)

- Theories with different vertices:

ϕ^3 : 

ϕ^6 : 

Evaluation of Feynman diagrams

- In general a difficult task – not always possible analytically. The complication grows with the loop level.

- Useful tools for Feynman integrals:

– Feynman parametrization:

$$\frac{1}{a_1 \dots a_n} = \frac{1}{(n-1)!} \int_{\{x_j \geq 0\}} \prod_{j=1}^{n} dx_j \, \frac{\delta(x_1 + \dots + x_n - 1)}{(x_1 a_1 + \dots + x_n a_n)^n}$$

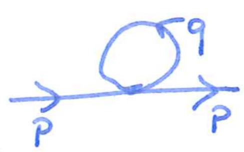
– Schwinger parametrization

$$\frac{1}{a^n} = \frac{1}{(n-1)!} \int_0^\infty du \, u^{n-1} e^{-ua}$$

– Momentum integrals:

$$\int \prod_{i=1}^{D} dp_i \, e^{-u \sum_{i=1}^{D} p_i^2} = \left(\frac{\pi}{u}\right)^{D/2}$$

- Example: one-loop two-point function



Using Feynman rules it evaluates to:

$$-\frac{\lambda}{2} \frac{1}{(p^2+m^2)^2} \underbrace{\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2}}_{I_2}$$

· There are many ways to evaluate I_2, e.g.

$$(q^2+m^2)^{-1} = \int_0^\infty e^{-u(q^2+m^2)}\,du$$

Then

$$I_2 = \int_0^\infty du \int \frac{d^Dq}{(2\pi)^D}\, e^{-u(q^2+m^2)}$$

$$= \int_0^\infty du\, \frac{\pi^{D/2}}{(2\pi)^D}\, u^{-D/2}\, e^{-um^2}$$

After the change of variable : $u \to u m^{-2}$

$$I_2 = \frac{\pi^{D/2}}{(2\pi)^D}\, \Gamma(1-D/2)\, m^{D-2}$$

where

$$\Gamma(t) \equiv \int_0^\infty x^{t-1} e^{-x}\, dx \quad , \quad \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N}$$

Then, up to one-loop:

$$\tilde{G}^{(2)}(p) = \frac{1}{(p^2+m^2)} + \frac{-\lambda}{2}\, \frac{1}{(p^2+m^2)^2}\, \frac{\pi^{D/2}}{(2\pi)^D}\, \Gamma(1-D/2)\, m^{D-2}$$

· The answer we found makes sense for non-integer values of D as well as for integer D smaller than 2. It reflects the fact that the integral

$$\int \frac{d^Dp}{p^2} \quad \text{converges only for } D<2.$$

For $D>2$ this integral is divergent and this kind of divergence is called UV divergence $\to$ it comes from the region where p is large (equivalently, when the distance between points is small)

- Ways to regulate UV divergencies:
  - introduce a cut-off Λ and integrate only over the region

$$|p| < \Lambda$$

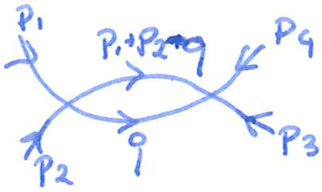
In the lattice field theory: $\Lambda \sim a^{-1}$

In two dimensions (at the boundary of convergence) the behavior is logarithmic in Λ (critical dimension)

$$I_2 \overset{D=2}{\sim} \frac{1}{2\pi} \log \frac{\Lambda}{m}$$

  - dimensional regularization:
    Evaluate the integral for $D<2$ and analytically continue the result

- Example: one-loop four-point function



with $p_1+p_2+p_3+p_4=0$

It evaluates to:

$$\frac{(-\lambda)^2}{2} \prod_{j=1}^{4} \frac{1}{p_j^2+m^2} \underbrace{\int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)((q-p_1+p_2)^2+m^2)}}_{I_4}$$

Using Feynman parametrization

$$I_4 = \int_0^1 dx \int \frac{d^Dq}{(2\pi)^D} \frac{1}{\left(x(q^2+m^2)+(1-x)((q-p_1+p_2)^2+m^2)\right)^2}$$

• The denominator:

$$q^2 - 2(1-x)(p_1+p_2)\cdot q + (1-x)(p_1+p_2)^2 + m^2$$

$$= (q')^2 + x(1-x)(p_1+p_2)^2 + m^2 \quad \text{with } q' = q - (1-x)(p_1+p_2)$$

Then:

$$I_4 = \int_0^1 dx \int \frac{d^D q'}{(2\pi)^D} \frac{1}{\left[(q')^2 + x(1-x)(p_1+p_2)^2 + m^2\right]^2}$$

$$= \int_0^1 dx \int_0^\infty du\, u\, e^{-u\left(x(1-x)(p_1+p_2)^2+m^2\right)} \int \frac{d^D q'}{(2\pi)^D} e^{-u(q')^2}$$

$$= \frac{\pi^{D/2}}{(2\pi)^D} \int_0^\infty du\, u^{1-D/2} e^{-u} \int_0^1 \left(x(1-x)(p_1+p_2)^2+m^2\right)^{D/2-2} dx$$

$$= \frac{\pi^{D/2}}{(2\pi)^D} \Gamma\left(2-\frac{D}{2}\right) \underbrace{\int_0^1 \left(x(1-x)(p_1+p_2)^2+m^2\right)^{D/2-2} dx}_{\text{can be done numerically. It is finite!}}$$

First pole is at $D=4$.

Renormalization

So far:

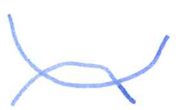
- we study correlators in perturbation theory:

$$\tilde{G}^{(N)}(p_1, \dots, p_N) = \sum_{n=0}^{\infty} \lambda^n \, \tilde{G}^{(N)}_n (p_1, \dots, p_N)$$

- for each N and n we have that $\tilde{G}^{(N)}_n(p_1, \dots, p_N)$ is a sum of integrals associated to connected Feynman diagrams with N external and n internal vertices

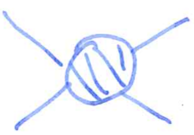
- in four dimensions some of these integrals diverge:

  - $N=2$:  $\simeq \displaystyle\int_{|q|\leq\Lambda} \frac{d^4q}{q^2} \sim \Lambda^2 + p^2 \log \Lambda + \text{finite}$

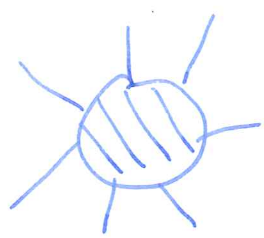
  - $N=4$:  $\simeq \displaystyle\int_{|q|\leq\Lambda} \frac{d^4q}{q^4} \sim \log \Lambda + \text{finite}$

- in general (for $D=4$)

$\sim \Lambda^2 + p^2 \log \Lambda + \text{finite}$



$\sim \log \Lambda + \text{finite}$



$\sim$ finite for $N > 4$

- General statement: most Feynman integrals are UV divergent for large enough D.

However: the perturbative expansion is written in powers of a quantity λ which is not directly measurable!

$\rightarrow$ there is no physical requirement for the coefficients in the expansion to be well-defined.

- The renormalization procedure attempts to make sense from this nonsense.

- General strategy:

① Relabel the fields $\phi \to \phi_0$ and the parameters

$$m \to m_0 \quad , \lambda \to \lambda_0$$

Similar for correlators: $G^{(N)} \to G_0^{(N)}$ (not to be confused with the free theory)

② Understand exactly where the divergences occur

③ Regularize the theory (make all Feynman integrals finite), e.g. cut off $|p| < \Lambda$ or dimensional regularization

④ Decide which quantities are physically measurable and compute them as a power series in bare parameter λ_0

Example. $m \xrightarrow{\lambda_0 \to 0} m_0$, $\lambda \xrightarrow{\lambda_0 \to 0} \lambda_0$