⑤ Try to eliminate m_0 and λ_0 in favor of m and λ in all physical quantities

⑥ If the resulting expressions have a finite limit as the regulator is removed, e.g. $\Lambda \to \infty$ or $D \to 4$, the theory is renormalizable → we can express all measurable observables in terms of a set of others.

Renormalization of ϕ^4

① Relabeling:

$$S = \int \left(\frac{1}{2} (\partial \phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{\lambda_0}{4!} \phi_0^4 \right) d^D x$$

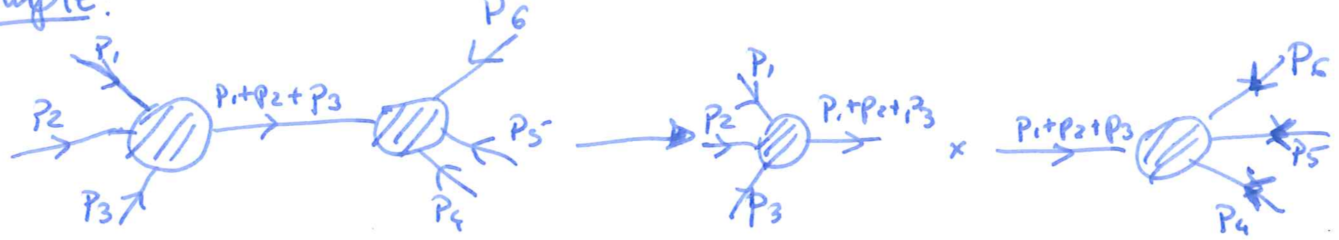
and
$$G_0^{(N)}(y_1, \dots, y_N) = \langle \phi_0(y_1) \dots \phi_0(y_N) \rangle$$

② What is the origin of divergencies?

Short answer: Loop integrals

Parts which are tree-like are problem-free

Example:



If both parts are finite then also their product is finite → the original diagram is finite

- We need to focus only on diagrams which cannot be divided into two disjoint parts with a one line cut $\longrightarrow$ 1PI diagrams

  one-particle-irreducible

- External propagators are always finite $\longrightarrow$ we can truncate them

- At the end we study

$$\Gamma_0^{(N)}(p_1, \dots, p_N) = \prod_{j=1}^{N} \tilde{G}_0^{(2)}(p_j)^{-1}\, \tilde{G}_0^{(N)}(p_1, \dots, p_N)\Big|_{1PI}$$

vertex functions

Example:

$$\Gamma_0^{(2)}(p) = \tilde{G}_0^{(2)}(p)^{-1}$$

- Counting divergences

- Any diagram depends on a number of loop momenta $(k_1, \dots, k_l)$, where l is the number of loops.

- UV divergences come from the region where some of k_j's are large $\rightarrow$ focus on the region where

$$k_j \sim K \leftarrow \text{large} \quad \forall_j$$

For l-loop diagram with P propagators, the divergence is:

$$\delta = l \cdot D - 2P \quad \text{(superficial degree of divergence)}$$

Divergences

If $\delta \geq 0$ – diagram primitively divergent

If $\delta < 0$ – not clear: there still can be divergences coming from regions where some k_j's are large while others are not

More refined analysis (diagram independent)

Calculate mass dimensions:

$$[p] = 1, \quad [x] = -1, \quad [S] = 0$$

$$\Rightarrow \quad [\phi] = \frac{D-2}{2} \quad \text{and} \quad [\lambda_0] = 4 - D$$

Correlators in position space:

$$[G_0^{(N)}] = \frac{N(D-2)}{2}$$

In momentum space:

$$[\tilde{G}_0^{(N)}] = \underbrace{D}_{\substack{\text{delta} \\ \text{function}}} - \underbrace{N \cdot D}_{\text{integrations}} + N\frac{D-2}{2}$$

and

$$[\Gamma_0^{(N)}] = N + D - N\frac{D}{2}$$

When expanded in powers of λ_0:

$$\Gamma_0^{(N)} = \sum_{n=0}^{\infty} \Gamma_{0,n}^{(N)} \lambda_0^n$$

given by a sum of 1PI Feynman diagrams

Divergence of $\Gamma_{0,n}^{(N)}$:

$$\delta = D + N\left(1 - \frac{D}{2}\right) + n(D-4)$$

If $\delta \geq 0 \rightarrow$ primitively divergent

We distinguish three cases:

- $D<4$: $[\Gamma^{(2)}_{0,n}] = 2 + n(D-4)$ and $[\Gamma^{(N)}_{n}] < 0$ for $N>2$

  $\rightarrow$ only a finite number of diagrams is divergent, namely, ones contributing to $\Gamma^{(2)}_{n}$ up to the order $n \leq \frac{2}{4-D}$

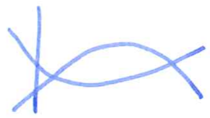
- $D=4$: $[\Gamma^{(2)}_{0}] = 2$, $[\Gamma^{(4)}_{0}] = 0$ are primitively divergent to all orders.

  Critical dimension of the theory (coupling dimensionless)

- $D>4$: all $\Gamma^{(N)}_{0}$ are primitively divergent if evaluated to sufficiently high-order (non-renormalizibity)

<u>Comment</u>: The critical dimension and the number of primitively divergent $\Gamma^{(N)}_{0}$ at the critical dimension depend on a theory.

• What about other divergences (not primitive)

Example:  $\leftarrow$ not divergent for $D=4$ but contains a divergent subgraph

- this kind of divergences occur in subdiagrams at lower order in λ_0 than the whole diagram

- we implement the renormalization procedure order-by-order: these divergences are already dealt with.

(3) Regularization: every regulator introduces a mass scale

- cut-off Λ $\rightarrow$ $[\Lambda]=1$
- dimensional regularization: $D=4-2\epsilon$, $\epsilon>0$ small

$$[\lambda_0]=2\epsilon$$

Introduce a dimensionless coupling constant

$$\underbrace{g_0}_{[g_0]=0} = \underbrace{\mu^{-2\epsilon}}_{[\mu]=1} \underbrace{\lambda_0}_{[\lambda_0]=2\epsilon}$$

(4) Mass renormalization

- We have already evaluated $\tilde{G}_0^{(2)}$ to one-loop

$$\tilde{G}_0^{(2)}(p) = \frac{1}{p^2+m_0^2} + \frac{-\lambda_0}{2}\frac{1}{(p^2+m_0^2)^2} I_2 + \mathcal{O}(\lambda_0^2)$$

with $I_2 = \frac{\pi^{D/2}}{(2\pi)^D}\Gamma(1-D/2)\, m_0^{D-2}$

- Then the two-point vertex function:

$$\Gamma_0^{(2)}(p) = \tilde{G}_0^{(2)}(p)^{-1} = p^2+m_0^2+\frac{\lambda_0}{2}I_2+\mathcal{O}(\lambda_0^2)$$

The position of the pole of $\tilde{G}_0^{(2)}$, or the zero of $\Gamma_0^{(2)}$, do not occur at $p^2=-m_0^2$, but rather depends on λ_0.

We can define renormalized mass m^2 as

$$\Gamma_0^{(2)}(p^2=-m^2)=0$$

Since I_2 is independent of p:

$$m^2 = m_0^2+\frac{\lambda_0}{2}I_2+\mathcal{O}(\lambda_0^2)$$

- Note that $m^2 > m_0^2$ so in the massless limit ($m^2 \to 0$) m_0^2 is negative → it underlines the fact that m_0 is not a physical parameter.
- For higher orders $\Gamma_{0,n}^{(2)}(p)$ depends on p and then m is given only implicitly.

Field renormalization

- Assume $D=4$ → $\Gamma_0^{(2)}(p)$ diverges quadratically.

One can expand it around $p^2=-m^2$

$$\Gamma_0^{(2)}(p) = \underbrace{\Gamma_0^{(2)}(p^2=-m^2)}_{=0} + \underbrace{(p^2+m^2)\left.\frac{\partial\Gamma_0^{(2)}(p)}{\partial p^2}\right|_{p^2=-m^2}}_{\substack{\text{divergent, with the degree } \delta=0 \\ \to \text{ logarithmic divergence}}}$$

- How can we absorb this divergence?

→ $\Gamma_0^{(2)}$ is defined using $\langle\phi_0\phi_0\rangle$ but ϕ_0 is just an integration variable and it has no physical meaning.

Define a physical field:

$$\phi(x) = Z_\phi^{-1/2}\,\phi_0(x)$$

field renormalization constant

Then

$$G^{(N)}(y_1,\dots,y_N) \equiv \langle\phi(y_1)\dots\phi(y_N)\rangle = Z_\phi^{-N/2}\langle\phi_0(y_1)\dots\phi_0(y_N)\rangle$$

This implies:

$$\Gamma^{(N)} = Z_\phi^{N/2}\, \Gamma_0^{(N)}$$

- Z_ϕ is fixed by the requirement

$$\left.\frac{\partial \Gamma^{(2)}(p)}{\partial p^2}\right|_{p^2=-m^2} = 1$$

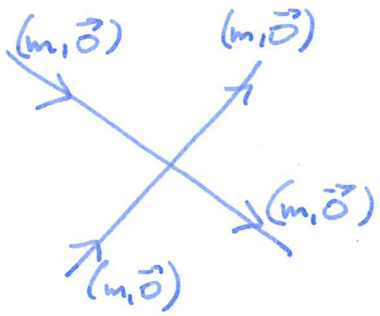
or equivalently

$$Z_\phi^{-1} = \left.\frac{\partial \Gamma_0^{(2)}(p)}{\partial p^2}\right|_{p^2=-m^2}$$

Coupling constant renormalization

Assume $D=4$. $\Gamma_0^{(4)}$ is primitively divergent to all orders. We define renormalized coupling constant

$$-\lambda \equiv \Gamma^{(4)}\big(\underset{\vec{p}=\vec{0}}{(m,\vec{0})},\, (m,\vec{0}),\, (-m,\vec{0}),\, (-m,\vec{0})\big)$$



This interaction can be measured in experiments and fixes λ.