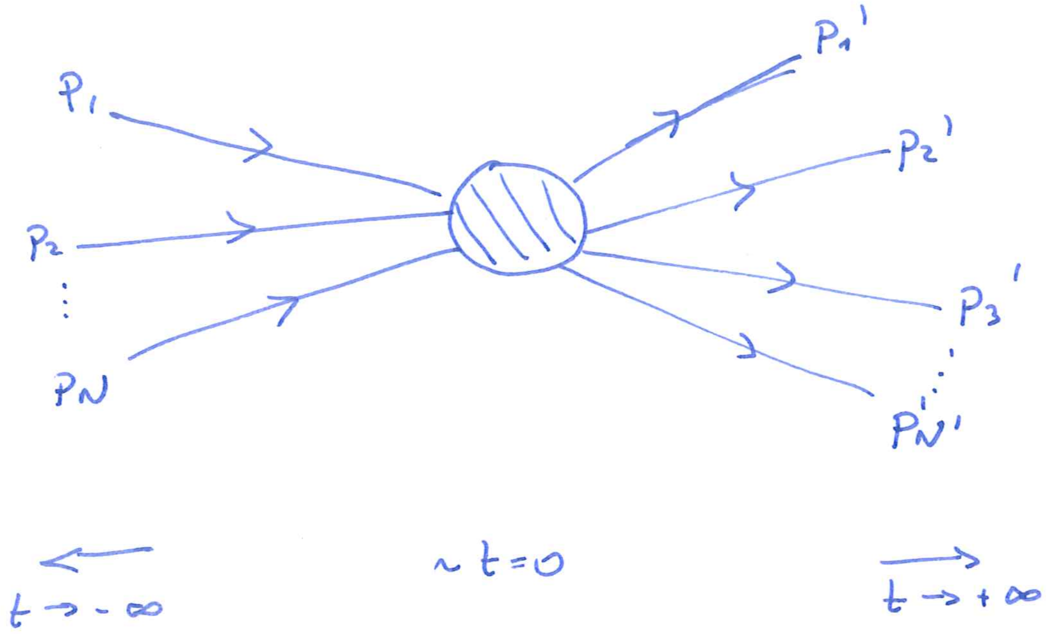
Scattering processes

In the following we try to model what goes on in accelerators.

We consider isolated wave packets prepared at $t \to -\infty$, allow them to interact around $t = 0$ and then measure what states come out as $t \to +\infty$.



In practice this situation is too hard to study. Instead we suppose that we have a large box and that as both $t \to \pm\infty$, the interaction is switched off. Then we can take the initial and final states to be described as elements of the free theory Hilbert space. We refer to them as <u>asymptotic states</u>.

Initially, in the Schrödinger picture we have a plane wave state $|p_1, \dots, p_N\rangle^{(s)}$. This evolves with the full Hamiltonian $\hat{H}^{(s)}$ and as $t \to +\infty$ we compute the transition amplitude:

$$\lim_{t\to\infty} {}^{(s)}\langle p_1', p_2', \dots, p_N' | e^{-i\hat{H}^{(s)}t} | p_1, \dots p_N\rangle^{(s)}$$

$$= {}^{(s)}\langle p_1', \dots, p_N' | \hat{S} | p_1, \dots, p_N\rangle^{(s)}$$

→ gives us a probability of scattering from one plane-wave state into another.

The operator $\hat{S}$ is called the <u>S-matrix</u>.

In the following we focus on the $2 \to 2$ scattering.

<u>Asymptotic states</u>

Free theory remainder:

$$|\vec{p}\rangle = \hat{a}^+(\vec{p})|0\rangle$$

with

$$\hat{a}^+(\vec{p}) = -i\int d^3\vec{x}\, e^{ipx}(-i\omega + \partial_0)\phi(x)$$

and $\omega = \sqrt{\vec{p}^{\,2} + m^2}$, $p^0 = \omega$

We normalized creation and annihilation operators such that:

$$[\hat{a}(\vec{p}), \hat{a}^+(\vec{p})] = (2\pi)^3\, 2\omega\, \delta^3(\vec{p} - \vec{p}\,')$$

• Vacuum state:

$$\overset{*}{a}(\vec{p})\,|0\rangle = 0 \quad \text{with} \quad \langle 0|0\rangle = 1.$$

This gives a Lorentz invariant normalization

$$\langle \vec{p}\,|\,\vec{p}\,'\rangle = (2\pi)^3\, 2\omega\, \delta^3(\vec{p}-\vec{p}\,')$$

• In free theory we can define a time-independent operator

$$a_i^{+} \equiv \int d^3\vec{p}\; f_i(\vec{p})\, \overset{*}{a}^{+}(\vec{p})$$

with

$$f_i(\vec{p}) \cong \exp\left(-\frac{(\vec{p}-\vec{p}_i)^2}{4\sigma^2}\right)$$

a particle localized in the momentum space near $\vec{p}_i$ and localized in the position space near the origin

• In the Schrödinger picture $a_i^+|0\rangle$ evolves with time

→ wave packet propagates and spreads out, the particle is localized far from the origin at $t \to \pm\infty$

• Two-particle state $a_i^+ a_j^+|0\rangle$ with $\vec{p}_i \neq \vec{p}_j$ is widely separated in the past

• In the interacting theory a_i^+ is not time-independent anymore → $a_i^+(t)$

A guess for a suitable initial state (Heisenberg picture)

$$|i\rangle = \lim_{t\to -\infty} a_1^+(t)\, a_2^+(t)\,|0\rangle$$

Similar for the final state:

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^{+}(t)\, a_{2'}^{+}(t)\, |0\rangle$$

Then the scattering amplitude is given by the overlap of the initial and final states:

$$\langle f|i\rangle = \langle 0| a_{1'}(+\infty)\, a_{2'}(+\infty)\, a_1^{+}(-\infty)\, a_2^{+}(-\infty)\, |0\rangle$$

$$= \langle 0| T\{ a_{1'}(+\infty)\, a_{2'}(+\infty)\, a_1^{+}(-\infty)\, a_2^{+}(-\infty) \}\, |0\rangle$$

operators are already time ordered.

- We can relate $a_i^{+}(+\infty)$ with $a_i^{+}(-\infty)$:

$$a_i^{+}(+\infty) - a_i^{+}(-\infty) = \int_{-\infty}^{+\infty} dt\, \partial_0\, a_i^{+}(t)$$

$$= -i \int d^3\vec{p}\, f_i(\vec{p}) \int d^4x\, \partial_0 \left(e^{ipx} (-i\omega + \partial_0)\, \phi(x) \right)$$

$$= -i \int d^3\vec{p}\, f_i(\vec{p}) \int d^4x\, e^{ipx} (\partial_0^2 + \omega^2)\, \phi(x)$$

$$= -i \int d^3\vec{p}\, f_i(\vec{p}) \int d^4x\, e^{ipx} (\partial_0^2 + \vec{p}^2 + m^2)\, \phi(x)$$

$$= -i \int d^3\vec{p}\, f_i(\vec{p}) \int d^4x\, e^{ipx} (\partial_0^2 - \overleftarrow{\nabla}^2 + m^2)\, \phi(x)$$

$$= -i \int d^3\vec{p}\, f_i(\vec{p}) \int d^4x\, e^{ipx} (\partial_0^2 - \vec{\nabla}^2 + m^2)\, \phi(x)$$

$$= -i \int d^3\vec{p}\, f_i(\vec{p}) \int d^4x\, \underbrace{(\partial_\mu \partial^\mu + m^2)\, \phi(x)}_{0 \text{ in free theory}}$$

but not zero in interacting theory

e.g. $\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$: $(\partial^2 + m^2)\phi = \frac{\lambda}{3!}\phi^3$

- Finally:

$$a_i^+(-\infty) = a_i^+(+\infty) + i\int d^3\vec{p}\, f_i(\vec{p}) \int d^4x\, e^{ipx}(\partial^2+m^2)\,\phi(x)$$

Similar for the annihilation

$$a_i(+\infty) = a_i(-\infty) + i\int d^3\vec{p}\, f_i(\vec{p})\, e^{-ipx}(\partial^2+m^2)\,\phi(x)$$

- Time ordering moves all annihilation operators to the right and all creation operators to the left and they annihilate the vacuum.

- In the remaining pieces we can take the limit $\sigma \to 0$ and then

$$f_i(\vec{p}) \longrightarrow \delta^3(\vec{p}-\vec{p}_i)$$

Particles are <u>on-mass-shell</u> $\rightarrow$ they have prescribed momenta which satisfy:

$$p_i^2 = m^2$$

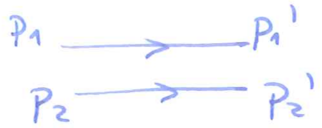
- We end up with the following formula called <u>LSZ reduction formula:</u>

$$\langle f|i\rangle = \int d^4x_1\, e^{ip_1x_1} \int d^4x_2\, e^{ip_2x_2} (\partial_1^2+m^2)(\partial_2^2+m^2)$$
$$\int d^4x_1'\, e^{-ip_1'x_1'} \int d^4x_2'\, e^{-ip_2'x_2'} (\partial_{1'}^2+m^2)(\partial_{2'}^2+m^2)$$
$$\langle 0|\, T\{\phi(x_1)\,\phi(x_2)\,\phi(x_1')\,\phi(x_2')\}\,|0\rangle$$

We expressed scattering amplitudes in terms of correlation functions! But we know already how to calculate them.

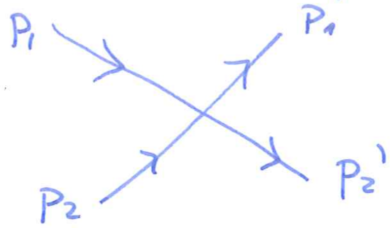
There are two types of diagrams we can encounter:

- disconnected e.g.



not interesting since there is no true scattering

- connected e.g.



- We split the S-matrix in two pieces

$$\underbrace{\hat{S} = \mathbb{1}}_{\text{sum over disconnected diagrams}} + i \underbrace{\hat{T}}_{\text{sum over connected diagrams}}$$

Then

$$i \langle p_1', \dots, p_{N'}' | \hat{T} | p_1, \dots, p_N \rangle$$

$$= \left. \frac{\tilde{G}^{(N+N')}(\{p_{j'}'\}, \{p_j\})_c}{\prod_{j'} \tilde{G}^{(2)}(p_{j'}') \prod_{j} \tilde{G}^{(2)}(p_j)} \right|_{p_{j'}'^2 = p_j^2 = m^2} \times \delta^4\Big(\sum_{j'} p_{j'}' - \sum_j p_j\Big)$$

The matrix elements of $i\hat{T}$ are given by the on-mass-shell values of the truncated renormalized connected $(N+N')$-point correlation functions

- Example:

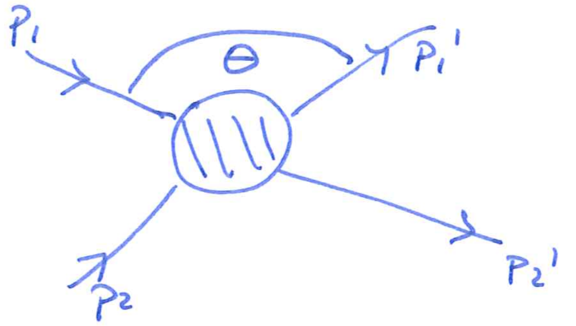
$$\langle p_1', p_2' | \hat{T} | p_1, p_2 \rangle = \lambda (2\pi)^4 \delta^4(p_1' + p_2' - p_1 - p_2) + \mathcal{O}(\lambda^2)$$

Cross-sections

- We do not measure scattering amplitudes in accelerators but rather cross-sections

- Elastic differential cross-section:

For $2 \to 2$ scattering in the centre of mass frame, all momenta are determined by the total centre of mass energy $E^2 = (p_1 + p_2)^2 = s$ and the scattering angle θ



Cross-section for this process

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{\lambda^2 + \mathcal{O}(\lambda^3)}{64\pi^2 s}$$

At the leading order it is independent of θ, but at higher-loop orders it is not true any more.

Unitarity and analyticity of the S-matrix

We split the S-matrix into connected and disconnected parts

$$\hat{S} = 1 + i\hat{T}$$

S-matrix is unitary:

$$\hat{S}\hat{S}^{+} = \hat{S}^{+}\hat{S} = 1$$

Then

$$\hat{T}^{+}\hat{T} = i(\hat{T}^{+} - \hat{T}) = 2\,\mathrm{Im}\,\hat{T}$$

Although we have seen that to lowest order $\hat{T} \sim \lambda$ is real, this shows that at higher orders it has an imaginary part.

- For the diagonal $2 \to 2$ scattering $(\vec{p}_1, \vec{p}_2) \to (\vec{p}_1, \vec{p}_2)$ we can insert a complete set of multiparticle states $|\vec{k}_1, \vec{k}_2, \ldots\rangle$ and get

$$LHS = \sum_{\{k_j\}} \langle \vec{p}_1, \vec{p}_2 | \hat{T}^{+} | \{\vec{k}_j\}\rangle \langle \{\vec{k}_j\} | \hat{T} | \vec{p}_1, \vec{p}_2\rangle$$

$\sim$ total cross-section for the scattering $2 \to n$, $n \geq 2$

Optical theorem:

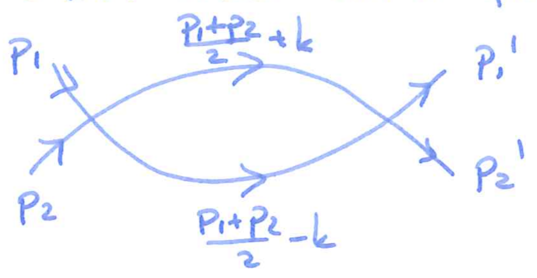
The imaginary part of the forward scattering amplitude is proportional to the total cross-section for the scattering $2 \to n$, $n \geq 2$

Unitarity and analyticity of the S-matrix

Example: ϕ^4

Matrix elements of $\hat{T}$ up to $\mathcal{O}(\lambda^2)$ order (in CM frame):

$iT \equiv$

$i\langle \vec{p}_1{}', \vec{p}_2{}' | \hat{T} | \vec{p}_1, \vec{p}_2 \rangle =$  $+$ 

$$= i\lambda + \frac{(i\lambda)^2}{2}\int \frac{dk^0 d^3\vec{k}}{(2\pi)^4}\frac{i^2}{\left((k^0-\frac{1}{2}E)^2-\vec{k}^2-m^2+i\epsilon\right)\left((k^0+\frac{1}{2}E)^2-\vec{k}^2-m^2+i\epsilon\right)}$$

where $E = p_1^0 + p_2^0$ is the total CM energy.

- Let us focus on the k^0 integral:

The poles of the integrand are at:

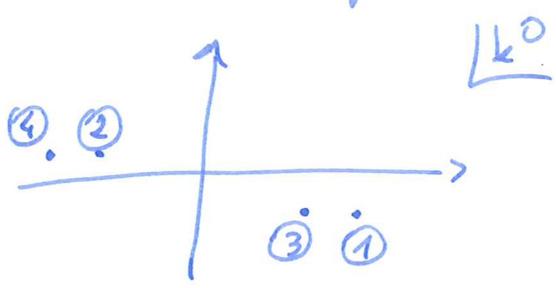
(1) $k^0 = \frac{1}{2}E + \sqrt{\vec{k}^2+m^2} - i\epsilon$

(2) $k^0 = \frac{1}{2}E - \sqrt{\vec{k}^2+m^2} + i\epsilon$

(3) $k^0 = -\frac{1}{2}E + \sqrt{\vec{k}^2+m^2} - i\epsilon$

(4) $k^0 = -\frac{1}{2}E - \sqrt{\vec{k}^2+m^2} + i\epsilon$

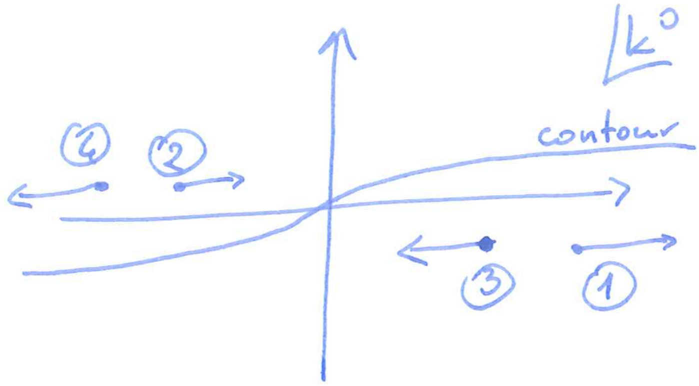
For $E < 2m$ their position is:



In that case we can complete the contour and we find that the contribution to T is real.

It is also analytic in E since we can always move the contour away from the poles and the integral is uniformly convergent

However, for $E > 2m$ the poles collide

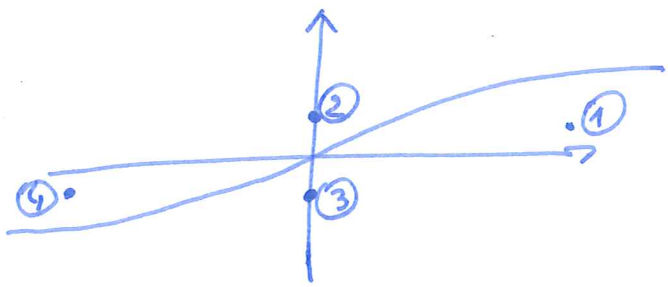


The arrows indicate how the positions of poles change when we increase E.

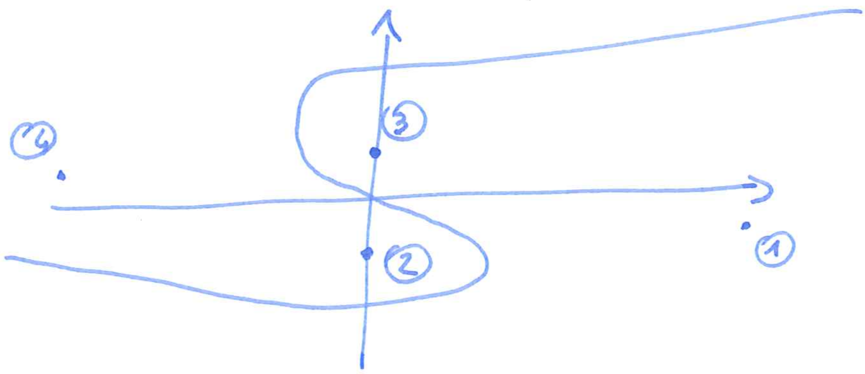
In order to understand the implications of pole colliding, let us study the analytic structure of this integral as a function of the energy E.

- We study the case when energy has a small imaginary part $E \to E + i\delta$, $|\delta| > \epsilon$

For $\delta \to 0^+$ we get



For $\delta \to 0^-$ we get

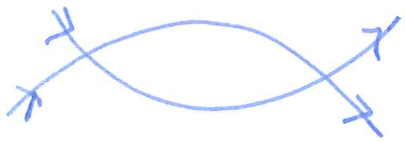
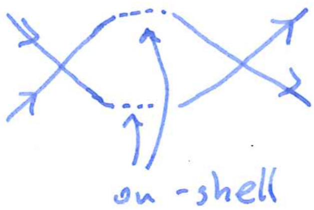


Unitarity and analyticity of the S-matrix

The discontinuity $\hat{T}(E+i\delta) - \hat{T}(E-i\delta)$ is imaginary and it is given by the values of the integrand where the poles collide:

$$\frac{\lambda^2}{2} \int \underbrace{\frac{dk^0\, d^3\vec{k}}{(2\pi)^4}\, \delta^+\!\left((k^0+\tfrac{1}{2}E)^2 - \vec{k}^2 - m^2\right) \delta^+\!\left((k^0-\tfrac{1}{2}E)^2 - \vec{k}^2 - m^2\right)}_{d\Pi},$$

This corresponds to inserting a complete set of intermediate (on-shell) states – only 2-particle states at this order.

$2\,\mathrm{Im}$  $= \int d\Pi$ 

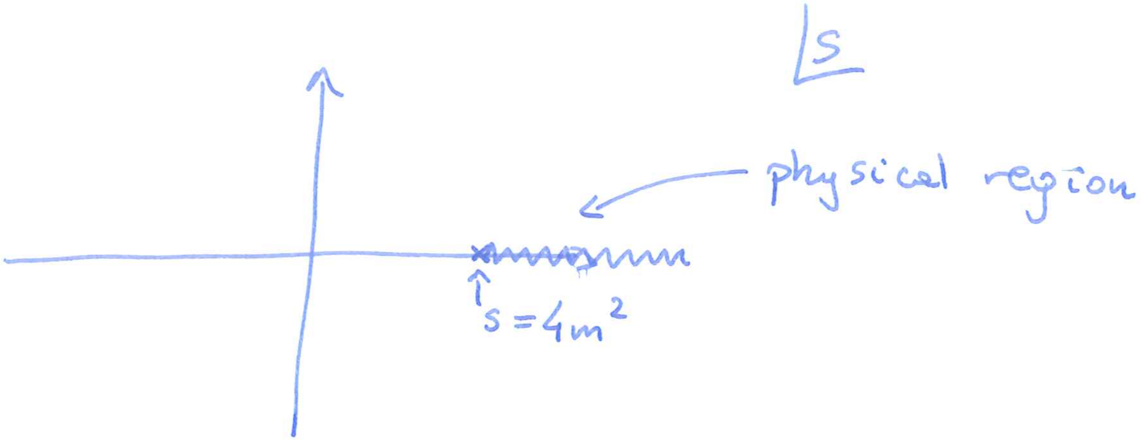
$= \frac{1}{2} \int d\Pi \;\Big|$  $\Big|^2$

The analyticity properties of the matrix elements of $\hat{T}$ we saw in this example are quite general and very powerful. Let us study it as a function of the relativistic invariant $s \equiv (p_1+p_2)^2 = E_{CM}^2$.

Then $T(s)$ is an analytic function of s, real on the real axis for $0 < s < 4m^2$, but it has a branch cut beginning at $s = 4m^2$

Unitarity and analyticity of the S-matrix

- Analytic structure in the s-plane:



The physical region is given by the values of T just above the cut. The point $s = 4m^2$ is called the 2-particle ~~the~~ threshold. For physical on-shell particles we must have $E_{CM} \geq 2m \Rightarrow s \geq 4m^2$. This corresponds to the value of s at which $2 \to 2$ scattering is possible.

- In general T is an analytic function of the relativistic invariants:

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_1')^2, \quad u = (p_1 - p_2')^2$$

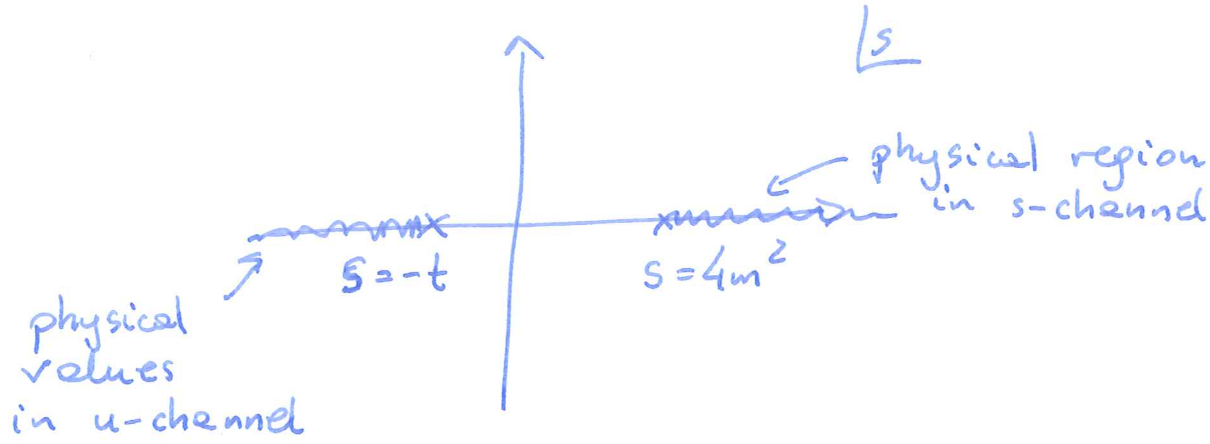
They satisfy:

$$s + t + u = 4m^2$$

After similar analysis one can find that there is also a branch cut starting at $u = 4m^2$ that is $s = -t$.

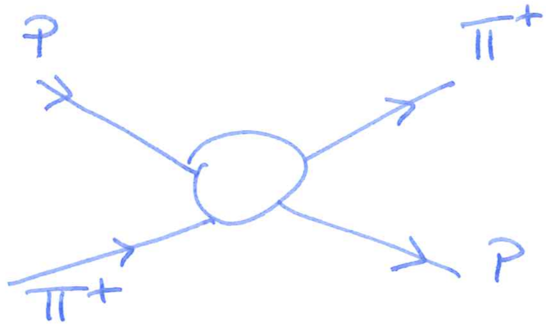
Unitarity and analyticity of the S-matrix

For (fixed) $t > 0$ the analytic structure of $T(s)$ is:

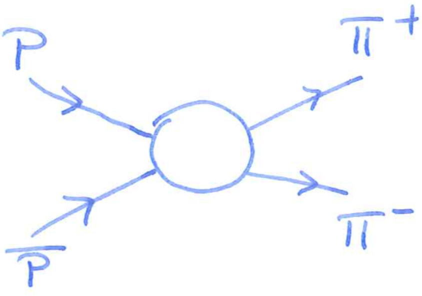


- This is important for the scattering of real particles. E.g.

s- channel :



u- channel



We can predict the cross-section for the second process if we know the amplitude for the first

→ it is given by an analytic continuation