Quantum Field Theory

Homework #1

Hand-in time and place (week 3):

Class	**Hand-in time**	**Hand-in place**	**Teaching Assistant**
Tuesday 15.30-17.00	Sunday 6pm	Mathematics†	Johan Henriksson
Friday 14.30-16.00	Tuesday noon	Mathematics†	Johan Henriksson
Thursday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi
Friday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi

† Mezzanine level in the Mathematical Institute (Andrew Wiles Building, Woodstock Road)

`DO NOT FORGET TO PUT THE NAME OF YOUR TEACHING ASSISTANT ON THE SHEET.`

(Problems with an asterisk ($*$) may be more difficult and are optional.)

1. **Scalar Field Theory** For the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\,, \qquad (1)$$

where ϕ is a real-valued scalar field:

(a) Derive the Klein-Gordon equation for ϕ from the least action principle.

(b) Find the momentum $\pi(x)$ conjugate to $\phi(x)$.

(c) Use $\pi(x)$ to calculate the Hamiltonian density $\mathcal{H}$.

(d) Using the transformation rules for scalar fields

$$\phi'(x') = \phi(x), \qquad \text{for } x'^\mu = \Lambda^\mu_\nu\, x^\nu\,, \qquad (2)$$

prove that the scalar field theory is invariant under the Lorentz transformations.

(e) Based on Noether's theorem, calculate the stress-energy tensor T^μ_ν of this field and the conserved charges associated with time and spatial transformations P^μ of this field.

(f) Using the Klein-Gordon equation show that $\partial_\mu T^\mu_\nu = 0$ for this field.

(g) Show that P_0 calculated in part (e) is the same as the total Hamiltonian, i. e. spatial integral of $\mathcal{H}$ calculated in part (c).

2. **Canonical Quantization of the complex scalar field**

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The Lagrangian of this theory is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{3}$$

(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi). \tag{4}$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is precisely the Klein-Gordon equation.

(b) Diagonalize the Hamiltonian H by introducing creation and annihilation operators. Show that the theory contains two sets of particles with mass m.

(c) Rewrite the conserved charge

$$Q = \frac{i}{2} \int d^3x (\phi^* \pi^* - \pi \phi)\,, \tag{5}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d*) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields $\phi_a(x)$, where $a = 1, 2$. Show that there are four conserved charges, one given by the generalization of the previous part, and other three given by

$$Q^i = \frac{i}{2} \int d^3x \sum_{a,b} \left(\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b \right)\,, \tag{6}$$

where σ^i are Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum (SU(2)).

Generalize these results to the case of n identical complex scalar fields.

3. **Free particle path integral**

(a) Consider the free particle path integral (with the mass $m = 1$ for simplicity)

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q(t) \exp\left[i \int_{t_i}^{t_f} \frac{\dot{q}^2}{2} dt \right] . \tag{7}$$

Write down a general path $q(t)$ as the sum of the classical path $q_c(t)$ (that is, motion at constant velocity) plus a Fourier series with coefficients $a_n, n \geq 1$.

(b) Show that the action for such a general path is

$$S = \frac{1}{2} \frac{(q_f - q_i)^2}{t_f - t_i} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2} \frac{(n\pi)^2}{t_f - t_i} a_n^2 . \tag{8}$$

(c) Perform the integral

$$\int da_n \, e^{iS} , \tag{9}$$

over a single Fourier mode.

(d) Write the entire path integral as a constant, depending only on $t_f - t_i$, times the classical action:

$$\int \prod_{n=1}^{\infty} da_n \, e^{iS} = c(t_f - t_i) \exp\left(\frac{i}{2} \frac{(q_f - q_i)^2}{t_f - t_i} \right) . \tag{10}$$

Does the constant have a finite value?

(e) The actual path integral measure contains a normalization constant γ

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q \, e^{iS} = \gamma \int \prod_{n=1}^{\infty} da_n \, e^{iS} , \tag{11}$$

such that the combination $\gamma \cdot c(t_f - t_i)$ is a finite number. The requirement that

$$\int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle = \langle q_f, t_f | q_i, t_i \rangle , \tag{12}$$

implies a relation between $\gamma \cdot c(t_f - t)$, $\gamma \cdot c(t - t_i)$ and $\gamma \cdot c(t_f - t_i)$. Find it and solve it. Hint: $\gamma \cdot c(\tau) \sim \tau^{-1/2}$.