Quantum Field Theory

Homework #2

Hand-in time and place (week 5):

Class	Hand-in time	Hand-in place	Teaching Assistant
Tuesday 15.30-17.00	Sunday 6pm	Mathematics†	Johan Henriksson
Friday 14.30-16.00	Tuesday noon	Mathematics†	Johan Henriksson
Thursday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi
Friday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi

† Mezzanine level in the Mathematical Institute (Andrew Wiles Building, Woodstock Road)

DO NOT FORGET TO PUT THE NAME OF YOUR TEACHING ASSISTANT ON THE SHEET.

4. **Wick theorem**

Recall the definition of the time ordering of fields

$$T\{\mathcal{O}(x_1)\mathcal{O}(x_2)\} = \begin{cases} \mathcal{O}(x_1)\mathcal{O}(x_2), & t_1 > t_2\,, \\ \mathcal{O}(x_2)\mathcal{O}(x_1), & t_2 > t_1 \end{cases}, \tag{1}$$

and the normal ordering of fields (all annihilation operators are moved to the right of all creation operators)

$$: a_i^\dagger a_j a_k^\dagger a_l := a_i^\dagger a_k^\dagger a_j a_l\,, \tag{2}$$

(a) Using the formula for the field written in terms of its Fourier modes

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{ip\cdot x} + a_p e^{-ip\cdot x})\,, \tag{3}$$

prove the Wick theorem for two fields:

$$T\{\phi(x_1)\phi(x_2)\} =: \phi(x_1)\phi(x_2) + \Delta_F(x_1 - x_2) :\,, \tag{4}$$

where $\Delta_F(x_1 - x_2)$ is the Feynman propagator.

(b) Prove the Wick theorem for n fields recursively.

(c) Using the Wick theorem evaluate the tree-level vacuum expectation value of

$$T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}\,, \tag{5}$$

for $n = 4, 5, 6$. Draw corresponding Feynman diagrams.

5. **Feynman diagrams and their symmetries**

(a) Compute symmetry factors W_G for diagrams in Figure 1.

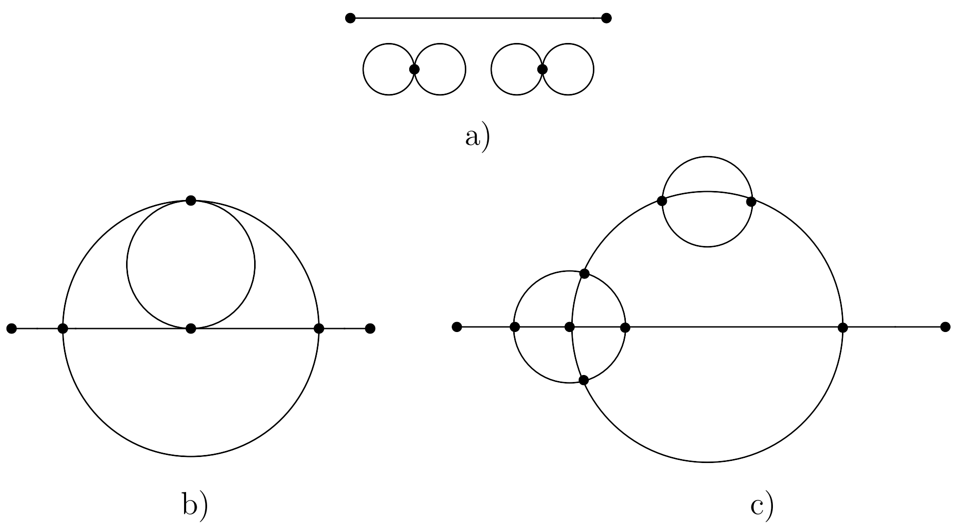


Figure 1:

(b) Confirm the formula

$$W_G = \frac{1}{2^{S+D} 3!^T N_{IVP}}\,, \tag{6}$$

where

- S is the number of self-connections;
- D is the number of double connections;
- T is the number of triple connections;
- N_{IVP} is the number of identical vertex permutations;

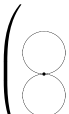
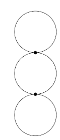
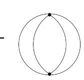
for all diagrams in Figure 1.

6. **Connected vs disconnected diagrams**

Consider the two-point function $\langle\phi(x)\phi(y)\rangle$ in the $\lambda\phi^4$ theory:

$$\langle\phi(x)\phi(y)\rangle = \frac{\langle 0|T\{\phi_I(x)\phi_I(y)\exp(-i\int_{-\infty}^{\infty} dt H_I(t))\}|0\rangle}{\langle 0|T\{\exp(-i\int_{-\infty}^{\infty} dt H_I(t))\}|0\rangle}\,. \tag{7}$$

(a) For the denominator of (7), identify different diagrams arising from an application of Wick's theorem up to the order λ^2. Confirm that to order λ^2, the combinatoric factors work out so that the denominator of (7) is given by the exponential of the sum of distinct vacuum bubble types

$$\langle 0|T\{\exp(-i\int_{-\infty}^{\infty} dt H_I(t))\}|0\rangle = \exp\Bigg($$  $+$  $+$  $+\dots\Bigg)\,.$ (8)

(b) Perform similar analysis for the numerator of (7) and prove, up to the order λ^2, that it is given by the sum of all connected diagrams times the exponential of the sum of all disconected diagrams. Prove that in order to calculate $\langle\phi(x)\phi(y)\rangle$ one needs to consider only contributions coming from connected diagrams.

7. **Feynman rules**

(a) Write the Feynman rules (in momentum space) for the theory of two real scalars of mass m and M with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}M^2\chi^2 + \frac{1}{2}\lambda\,\phi^2\chi\,. \tag{9}$$

Use plain and dashed lines for the ϕ and χ-propagator, respectively.

(b) Draw the tree and one-loop diagrams that contribute to the correlation functions $\langle\phi\phi\rangle$, $\langle\chi\chi\rangle$, $\langle\chi\rangle$ and $\langle\chi\phi\phi\rangle$, and write down explicit expressions for the one-loop diagrams, including the correct symmetry factors.