Quantum Field Theory

Homework #4

Hand-in time and place (week 8):

Class	Hand-in time	Hand-in place	Teaching Assistant
Tuesday 15.30-17.00	Sunday 6pm	Mathematics†	Johan Henriksson
Friday 14.30-16.00	Tuesday noon	Mathematics†	Johan Henriksson
Thursday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi
Friday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi

† Mezzanine level in the Mathematical Institute (Andrew Wiles Building, Woodstock Road)

`DO NOT FORGET TO PUT THE NAME OF YOUR TEACHING ASSISTANT ON THE SHEET.`

11. **Renormalisation of the ϕ^3 theory**

    Consider the theory of a massive real scalar field ϕ with interaction $\frac{1}{6}\lambda\phi^3$

    (a) What is the critical dimension D_c in which this theory is exactly renormalisable?

    (b) In D_c, which of the $\Gamma^{(N)}$ contain primitive divergencies, and how should these be made finite?

    (c) In the massless theory, using dimensional regularisation and minimal subtraction, work out the renormalised coupling constant at one loop in $D = D_c - 2\epsilon$. [This involves computing two one-loop diagrams, one for $\Gamma^{(3)}$ and also the field renormalisation from $\Gamma^{(2)}$ which in this theory has a contribution at one loop order.]

    (d) Work out the β-function to one loop.

12. **Asymptotic symmetry**

    Consider the following Lagrangian for a theory in Minkowski space with two scalar fields ϕ_1 and ϕ_2:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4) - \frac{2\kappa}{4!}(\phi_1^2\phi_2^2)\,. \qquad (1)$$

    Observe that, for the special value $\lambda = \kappa$, this Lagrangian has an $O(2)$ invariance rotating the two fields into each other.

(a) Working in four dimensions, find the β functions for two coupling constants λ and κ, to leading order in the coupling constants.

(b) Define $\nu = \kappa/\lambda$ and write the renormalisation group equation for ν. Show that, if $\nu < 3$ at a renormalisation point M, this ratio flows toward the condition $\nu = 1$ at large distances. Then the $O(2)$ internal symmetry appears asymptotically.

(c) Write the β functions for λ and κ in $4 - 2\epsilon$ dimensions. Show that there are nontrivial fixed points of the renormalisation group flow at $\nu = 0, 1, 3$. Which of these points is the most stable? Sketch the pattern of coupling constant flows.

13. **Renormalization group**

For the quantum field theory of a single massless field ϕ and a single dimensionless coupling constant g, at a regularization scale μ, consider the N-point vertex function $\Gamma^{(N)}(p_k, g(\mu), \mu)$. Let $\Gamma_0^{(N)}(p_k, \lambda_0)$ be the bare vertex function depending on the bare coupling λ_0 and on the regulator ϵ. The renormalized vertex function can be written in terms of the bare one as

$$\Gamma^{(N)}(p_k, g(\lambda_0, \mu), \mu) = Z_\phi(\lambda_0, \mu)^{N/2}\, \Gamma_0^{(N)}(p_k, \lambda_0)\,, \tag{2}$$

where $\Gamma^{(N)}$ is finite when the regulator ϵ is removed, namely, all divergencies are absorbed into the definition of the renormalized coupling $g(\lambda_0, \mu)$ and of the field renormalization $Z_\phi(\lambda_0, \mu)$.

The Callan-Symanzik equation for a generic N-point vertex function $\Gamma^{(N)}$ is

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \frac{N}{2}\gamma(g)\right)\Gamma^{(N)}(p_k, g, \mu) = 0\,, \tag{3}$$

where

$$\beta(g) = \mu\frac{\partial}{\partial\mu}g(\mu)\Big|_{\lambda_0}, \qquad \gamma(g) = \mu\frac{\partial \log Z_\phi}{\partial\mu}\Big|_{\lambda_0}. \tag{4}$$

(a) Assuming that the Callan-Symanzik equation holds for the bare vertex function $\Gamma_0^{(N)}$, namely for $\Gamma^{(N)}(p_k, g_0 = \lambda_0\mu^{-\epsilon}, \mu) = \Gamma_0^{(N)}(p_k, \lambda_0)$, show that

$$\beta_0(g_0) = -\epsilon\, g_0, \qquad \gamma_0(g_0) = 0\,. \tag{5}$$

(b) Assume that $g(\lambda_0, \mu) = g(g_0)$ and $Z_\phi(\lambda_0, \mu) = Z_\phi(g_0)$ and show that the Callan-Symanzik equation holds also for $\Gamma^{(N)}$ with coefficients

$$\beta(g(g_0)) = \beta_0(g_0)\frac{\partial g}{\partial g_0}, \qquad \gamma(g(g_0)) = \gamma_0(g_0) + \frac{\beta_0(g_0)}{Z_\phi(g_0)}\frac{\partial Z_\phi}{\partial g_0}. \quad (6)$$

(c) Consider the perturbative expansion of the coupling constant and of the field renormalization of the form

$$g(g_0) = g_0 + g_0^2\left(\frac{a_1}{\epsilon} + a_2 + a_3\epsilon + \dots\right), \quad (7)$$

$$Z_\phi(g_0) = 1 + g_0\left(\frac{z_1}{\epsilon} + z_2 + z_3\epsilon + \dots\right). \quad (8)$$

Compute $\beta(g)$ and $\gamma(g)$ and check that they are finite in the absence of regulator.

(d) Now consider the next perturbative order for the coupling constant and focus on the leading order in ϵ, that is

$$g(g_0) = g_0 + g_0^2\frac{a_1}{\epsilon} + g_0^3\frac{b_1}{\epsilon^2}. \quad (9)$$

Compute $\beta(g)$ to order g^3 and $\frac{1}{\epsilon}$ and show that its finiteness implies that the coefficients entering the two loop corrections are dependent on the lower order ones.

14. **Scattering matrices**

Consider the interacting $\lambda\phi^4$ theory with the Lagrangian density

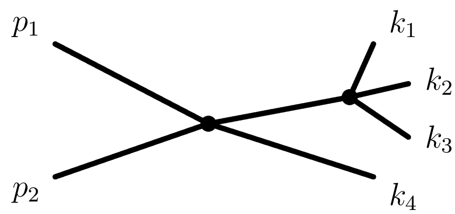
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4. \quad (10)$$

In the interaction picture the S-matrix is given by

$$S = T\left\{\exp\left(i\int d^4x\,\mathcal{L}_{int}(x)\right)\right\} \quad (11)$$

(a) Consider the scattering of four incoming particles with momenta k_1, k_2, k_3 and k_4 to two outgoing particles with momenta p_1 and p_2. Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle p_1, p_2|S|k_1, k_2, k_3, k_4\rangle$:

What is the order in perturbation expansion at which this diagram appear?



(b) By doing the position-space integrals show that the contribution can be written as

$$\frac{-i\lambda^2}{(p_1+p_2-k_4)^2-m^2+i\epsilon}(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2-k_3-k_4)\,. \quad (12)$$

What is the physical meaning of the delta-function?

(c) There are three more connected diagrams with a topology similar to that in part (a) and six further connected diagrams with a different topology all of which contribute at the same order in λ. Draw one further diagram of each type and use the momentum space Feynman rules to write down the contribution of each to $\langle p_1, p_2|S|k_1, k_2, k_3, k_4\rangle$.