Warm-up exercises

① Lagrangian:

$$\mathcal{L} = \frac{m}{2}(\dot{x}^i)^2 + q A_i(x)\dot{x}^i - q\,\phi(x)$$

a) Canonical momenta:

$$p^i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = m\dot{x}^i + q A_i(x)$$

b) Equations of motion:

$$\frac{\partial \mathcal{L}}{\partial x^j} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^j}$$

$$LHS = q\frac{\partial A_i(x)}{\partial x^j}\dot{x}^i - q\frac{\partial \phi(x)}{\partial x^j}$$

$$RHS = \frac{d}{dt}(m\dot{x}^j + q A_j(x)) = m\ddot{x}^j + q\dot{A}_j(x) + q\frac{\partial x^i}{\partial t}\frac{\partial A_j}{\partial x^i}$$

$$\Rightarrow m\ddot{x}^j = q\left(-\dot{A}_j(x) - \frac{\partial \phi(x)}{\partial x^j} + \frac{\partial A_i}{\partial x_j}\dot{x}^i - \dot{x}^i\frac{\partial A_j}{\partial x^i}\right)$$

Define:

$$E = -\nabla\phi(x) - \dot{A}(x) \Rightarrow E_i = -\frac{\partial}{\partial x^i}\phi(x) - \dot{A}_i(x)$$

and

$$B = \nabla \times A$$

Then $(\dot{x}\times B)_j = (\dot{x}\times(\nabla\times A))_j = \epsilon_{jik}\,\dot{x}^i(\nabla\times A)_k$

$$= \epsilon_{jik}\,\dot{x}^i\,\epsilon_{klm}\frac{\partial A_m}{\partial x^l} = (\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il})\dot{x}^i\frac{\partial A_m}{\partial x^l}$$

$$= \dot{x}^m\frac{\partial A_m}{\partial x^j} - \dot{x}^l\frac{\partial A_j}{\partial x^l}$$

Finally: $m\ddot{x}^j = q(E_j + (\dot{x}\times B)_j)$

$$\Rightarrow \quad m\ddot{x} = \underbrace{q(E + v\times B)}_{\text{Lorenz force}}$$

Warm-up exercises

c) Hamiltonian

$$H = \sum_i \dot{x}^i p^i - \mathcal{L} =$$

$$= \dot{x}^i \left(m\dot{x}^i + q A_i(x)\right) - \frac{m}{2}(\dot{x}^i)^2 - q A_i(x)\dot{x}^i + q\,\phi(x)$$

$$= \frac{1}{2} m (\dot{x}^i)^2 + q\,\phi(x)$$

For the free particle we set $q=0$ in all formulas.

(2) Hamiltonian for harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \longrightarrow \hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2 \hat{x}^2}{2}$$

a) Introduce :

$$a^+ = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} + i\hat{p})$$

$$a = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} - i\hat{p})$$

Then:

$$a^+ a = \frac{1}{2\omega}(\omega^2 \hat{x}^2 + p^2) - \frac{1}{2}$$

and

$$H = \frac{1}{2\omega}(\omega^2 \hat{x}^2 + \hat{p}^2) = (a^+ a + \tfrac{1}{2})\,\omega$$

Spectrum of H:

- ground state: $a|0\rangle = 0$, $H|0\rangle = \frac{1}{2}|0\rangle$
- excited states: $|n\rangle = (a^+)^n|0\rangle$, $H|n\rangle = (n+\frac{1}{2})|n\rangle$

Warm-up exercises

b) Expectation values in $|n\rangle$:

$$\langle N\rangle = \langle n|N|n\rangle = n\,\langle n|n\rangle = n$$

$$\langle x\rangle = \langle n|\hat{x}|n\rangle = \langle n|\frac{1}{\sqrt{2\omega}}(a+a^{+})|n\rangle = 0$$

$$\langle p\rangle = \langle n|\hat{p}|n\rangle = \langle n|i\sqrt{\frac{\omega}{2}}(a^{+}-a)|n\rangle = 0$$

c) Variances:

$$\Delta N = \langle N^2\rangle - \langle N\rangle^2$$

$$\langle N^2\rangle = \langle n|N^2|n\rangle = \langle n|n\rangle\cdot n^2 = n^2$$

Then:

$$\Delta N = n^2 - n^2 = 0.$$

$$\Delta x = \langle x^2\rangle - \langle x\rangle^2$$

$$\langle x^2\rangle = \frac{1}{2\omega}\langle n|(a+a^{+})^2|n\rangle = \frac{1}{2\omega}\langle n|a^{+}a+a^{+}a+1|n\rangle$$

$$= \frac{1}{2\omega}\langle n|2n+1|n\rangle = \frac{2n+1}{2\omega} = (n+\tfrac{1}{2})\omega$$

Similar:

$$\langle p^2\rangle = \omega(n+\tfrac{1}{2})$$

Then $\quad \Delta x\cdot\Delta p = (n+\tfrac{1}{2})$

Restoring the $\hbar$ dependence:

$$\Delta x\cdot\Delta p = (n+\tfrac{1}{2})\hbar > \frac{\hbar}{2} \quad \checkmark$$

Warm-up exercises

d) Coherent state:

$$|\alpha\rangle = e^{\alpha p}|0\rangle = e^{\tilde{\alpha}(a^+ - a)}|0\rangle \quad \text{with } \tilde{\alpha} = i\alpha\sqrt{\frac{\omega}{2}}$$

Prove that $|\alpha\rangle$ is eigenvector for a:

$$a|\alpha\rangle = a\, e^{\tilde{\alpha}(a^+ - a)}|0\rangle = a \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{n!}(a^+ - a)^n|0\rangle$$

Commutator:

$$a\,(a^+ - a)^n = a(a^+ - a)(a^+ - a)^{n-1} = (a^+ a - a a + 1)(a^+ - a)^{n-1}$$

$$= (a^+ - a)^{n-1} + (a^+ - a)\, a\, (a^+ - a)^{n-1} = n\,(a^+ - a)^{n-1} + (a^+ - a)^n a$$

Then:

$$a|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{n!}\Big(\underbrace{(a^+ - a)^n a|0\rangle}_{=0} + n\,(a^+ - a)^{n-1}|0\rangle\Big)$$

$$= \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{(n-1)!}(a^+ - a)^{n-1}|0\rangle = \tilde{\alpha} \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{n!}(a^+ - a)^n|0\rangle =$$

$$= \tilde{\alpha}\, e^{\tilde{\alpha}(a^+ - a)}|0\rangle = \tilde{\alpha}\,|\alpha\rangle$$

e) We study the time dependent operators:

$$\hat{x}(t) = e^{-iHt}\,\hat{x}\,e^{iHt}$$

First, let us expand $|\alpha\rangle$ in energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

$$\rightarrow \quad a|\alpha\rangle = \sum_{n=0}^{\infty} c_n\, a|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle$$

$$\rightarrow \quad \langle m|a|\alpha\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,\langle m|n-1\rangle$$

$$\tilde{\alpha}\, c_m = c_{m+1}\sqrt{m+1}$$

$$\rightarrow \quad c_n = \frac{\tilde{\alpha}^n}{\sqrt{n!}}\,\mathcal{N}(\tilde{\alpha})$$

Warm-up exercises

The normalisation can be found from

$$\langle\alpha|\alpha\rangle = 1 \quad \Rightarrow \quad \mathcal{N}(\tilde{\alpha}) = e^{-\frac{|\tilde{\alpha}|^2}{2}}$$

Finally :

$$|\alpha\rangle = e^{-\frac{|\tilde{\alpha}|^2}{2}} \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{\sqrt{n!}} |n\rangle$$

We can define the state:

$$|\alpha(t)\rangle = e^{iHt}|\alpha\rangle = e^{i\frac{t}{2}} e^{i\omega a^+ a t} e^{-\frac{|\tilde{\alpha}|^2}{2}} \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{\sqrt{n!}} |n\rangle$$

$$= e^{i\frac{t}{2}} e^{-\frac{|\tilde{\alpha}|^2}{2}} \sum_{n=0}^{\infty} \frac{(\tilde{\alpha} e^{int\omega})^n}{\sqrt{n!}} |n\rangle = e^{i\frac{t}{2}} \underbrace{|\alpha e^{i\omega t}\rangle}_{\text{coherent state}}$$

We can calculate now:

$$\langle\alpha|\hat{x}(t)|\alpha\rangle = \langle\alpha(t)|\hat{x}|\alpha(t)\rangle = \langle\alpha(t)|a^+ + a|\alpha(t)\rangle \frac{1}{\sqrt{2\omega}}$$

$$= \frac{1}{\sqrt{2\omega}} (\tilde{\alpha}^* + \tilde{\alpha}) \langle\alpha(t)|\alpha(t)\rangle = \frac{1}{\sqrt{2\omega}} \operatorname{Re}\tilde{\alpha}$$

Similar :

$$\langle\alpha|p(t)|\alpha\rangle = \sqrt{\frac{\omega}{2}} \operatorname{Im}\tilde{\alpha}^*$$

and

$$\langle\alpha|N(t)|\alpha\rangle \underset{\text{since } [N,H]=0}{=} \langle\alpha|N|\alpha\rangle = \langle\alpha|a^+ a|\alpha\rangle = |\tilde{\alpha}|^2$$

Variances:

$$\langle\hat{x}(t)^2\rangle = \frac{1}{2\omega}\left((\tilde{\alpha}+\tilde{\alpha}^*)^2 + 1\right) \quad \text{and} \quad \Delta x = \frac{1}{\sqrt{2\omega}}$$

$$\langle\hat{p}(t)^2\rangle = \frac{\omega}{2}\left((\tilde{\alpha}-\tilde{\alpha}^*)^2 - 1\right) \qquad \Delta p = \sqrt{\frac{\omega}{2}}$$

Then:

$$\Delta x \cdot \Delta p = \frac{1}{2} \quad \checkmark$$

Saturates the Heisenberg principle

Warm-up exercises

③ $$S = -\alpha \int ds = -\alpha \int \sqrt{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}}\, d\tau$$

a) Lorentz transformation invariance:

$$X^\mu \to \Lambda^\mu{}_\rho X^\rho$$

$$S \to -\alpha \int \sqrt{\underbrace{\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma}_{\eta_{\rho\sigma}} \frac{\partial X^\rho}{\partial \tau} \frac{\partial X^\sigma}{\partial \tau}}\, d\tau \quad \checkmark$$

b) Reparametrisation invariance:

$$\tau \to \tau'(\tau), \qquad \frac{\partial X^\mu(\tau)}{\partial \tau} \to \frac{\partial X^\mu(\tau')}{\partial \tau'} \frac{\partial \tau'}{\partial \tau}$$

$$d\tau \to \frac{\partial \tau}{\partial \tau'} d\tau'$$

$$S \to -\alpha \int \sqrt{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau'} \frac{\partial X^\nu}{\partial \tau'} \left(\frac{\partial \tau'}{\partial \tau}\right)^2}\, \frac{\partial \tau}{\partial \tau'} d\tau' \quad \checkmark$$

c) Momentum:

$$\mathcal{L} = -\alpha \sqrt{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}}$$

$$p^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -\frac{\alpha}{2} \frac{\frac{\partial}{\partial \dot{x}^\mu}\left(\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\right)}{\sqrt{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}}} = -\alpha \frac{\dot{x}^\mu}{\sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}$$

d) Non-relativistic limit

$$S \to -\alpha \int \sqrt{1 - (\dot{x}^i)^2}\, dt \cong -\alpha \int \left(1 - \frac{(\dot{x}^i)^2}{2} + \dots\right) dt$$

Then $\mathcal{L} = -\alpha + \dfrac{\alpha (\dot{x}^i)^2}{2} + \dots$

$$\Rightarrow \boxed{\alpha = m}$$