String Theory I: Problem Sheet 1

Hilary Term 2018

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1. Veneziano and Virasoro-Shapiro amplitudes

Before they were recognized as describing relativistic quantum-mechanical strings, a great deal of progress was made by studying the properties of scattering amplitudes as a part of the *dual model* program. In this problem, you will perform some analysis on the two most famous dual model amplitudes.

1. Consider elastic scattering of identical scalar particles with masses $\alpha' m^2 = -\alpha(0) = -1$. In terms of the Mandelstam variables

$$s = -(p_1 + p_2)^2 \ , \qquad t = -(p_1 + p_4)^2 \ , \qquad u = -(p_1 + p_3)^2 \ ,$$

find the expression for the scattering angle θ_s in center-of-momentum frame. Characterize the limit of high energy fixed-angle scattering in terms of Mandelstam variables. What does the Regge limit ($s \gg 1$, $t < 0$ fixed) look like in center-of-momentum frame?

2. The Veneziano amplitude for the scattering of open-string tachyons of mass $\alpha' m^2 = -1$ is given by

$$\mathcal{A}_V(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \ , \qquad \alpha(x) := 1 + \alpha' x \ .$$

Show that the Veneziano amplitude can also be defined by the series expansion

$$\mathcal{A}_V(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)(\alpha(s)+2)\cdots(\alpha(s)+n)}{n!} \frac{1}{\alpha(t)-n} \ .$$

Deduce that the Veneziano amplitude displays Dolan-Horn-Schmid duality.

3. Show that in the Regge limit, the Veneziano amplitude behaves according to[1]

$$\mathcal{A}(s,t) \sim \Gamma(-\alpha(t))(-\alpha(s))^{\alpha(t)} \ .$$

This is subtle: you must actually define the Regge limit so that s has a small imaginary part to avoid s-channel poles on the positive s axis. Can you justify this trick?

4. Show that in high-energy, fixed-angle scattering, the Veneziano amplitude behaves according to

$$\mathcal{A}(s,t) \sim F(\theta_s)^{-\alpha(s)} \ .$$

Find the function $F(\theta_s)$ and show that this behavior is exponentially soft. As in the previous part, you should give s a small imaginary part to avoid s-channel poles.

5. The Virasoro-Shapiro amplitude for the scattering of identical scalar tachyons of mass $\alpha' m^2 = -4$ is given by[2]

$$\mathcal{A}_{VS}(s,t,u) = \frac{\Gamma(-\alpha_c(s))\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))}{\Gamma(-\alpha_c(s)-\alpha_c(t))\Gamma(-\alpha_c(t)-\alpha_c(u))\Gamma(-\alpha_c(u)-\alpha_c(s))} \ , \qquad \alpha_c(x) := 1 + \frac{\alpha' x}{4} \ .$$

Find and argue the validity of an expression for $\mathcal{A}_{VS}(s,t,u)$ as a sum of, say, t- and u-channel exchange contributions (*i.e.*, as a sum of terms which have simple poles with respect to t or u with residues that are polynomials in s, as would arise from tree-level exchange diagrams in the t or u channels in field theory).

6. Find expressions for the Regge and high-energy, fixed-angle scattering limits of the Virasoro-Shapiro amplitude.

[1] Stirling's formula for the Gamma function should prove useful for this.

[2] Recall that $s+t+u = 4m^2 = -16/\alpha'$ for this amplitude. We write the u dependence to make crossing symmetry manifest.

2. Classical string dynamics

Although the analysis of the quantum string is quite a bit more involved than that of the classical string, it is important to remember that there are some points of contact between classical and quantum theories. In this exercise, you will analyze the classical bosonic string via the Nambu-Goto action, which is less suitable for quantization than the Polyakov action, but is a perfectly good (and equivalent) classical theory.

1. Recall the Nambu-Goto Lagrangian for the relativistic string,

$$\mathcal{L}_{\rm NG} = -T\sqrt{-h}\ ,$$

where h is the determinant of the induced metric on the worldsheet. Derive the equations of motion implied by this Lagrangian for both open and closed strings, and show that in both cases they imply

$$\partial_i K^i_\mu = 0\ , \qquad K^i_\mu := \frac{\delta\mathcal{L}}{\delta\left(\frac{\partial x^\mu(\xi)}{\partial\xi^i}\right)}\ ,$$

where ξ^i are worldsheet coordinates and $x^\mu(\xi)$ are space-time coordinates.

Using the equations of motion, show that the following quantity is conserved by the string dynamics,

$$P^\mu(\tau) = \int_0^\pi d\sigma\, K^{\mu\tau}(\sigma,\tau)\ ,$$

where σ and τ are spatial and temporal coordinates on the string worldsheet. What is the interpretation of this quantity?

2. Show that in conformal gauge, the following quantity is also conserved for both open and closed strings,

$$M^{\mu\nu} = \int_0^\pi d\sigma\,\Big(x^\mu(\sigma,\tau)K^{\nu\tau}(\sigma,\tau) - x^\nu(\sigma,\tau)K^{\mu\tau}(\sigma,\tau)\Big)\ .$$

3. Consider an open string in conformal gauge. Show that the end-points of the string move through spacetime at the speed of light.

4. Verify that the following is a solution of the equations of motion for the Nambu-Goto string in conformal gauge:

$$\begin{aligned}
x^0(\sigma,\tau) &= \frac{1}{2}\left(p+\frac{a^2}{p}\right)n\tau\ ,\\
x^1(\sigma,\tau) &= \frac{1}{2}\left(p-\frac{a^2}{p}\right)n\tau\ ,\\
x^2(\sigma,\tau) &= a\cos(n\sigma)\cos(n\tau),\\
x^3(\sigma,\tau) &= a\cos(n\sigma)\sin(n\tau),\\
x^\mu(\sigma,\tau) &= 0\ , \qquad\qquad \mu\geqslant 4\ .
\end{aligned}$$

Describe the motion of this string through spacetime. Find an analogous solution where the center of mass of the string is stationary. Find the relationship between spacetime energy and angular momentum for this family of solutions.

Please send comments and corrections to `christopher.beem@maths.ox.ac.uk`.