String Theory I: Problem Sheet 2

Hilary Term 2018

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1. Charge algebras for the classical string

In Hamiltonian mechanics, conserved charges are represented by functions on phase space that commute with the Hamiltonian, and the action of the symmetry corresponding to a charge on an observable is implemented by via the Poisson bracket. In this exercise, you will verify the algebra of the conserved charges for spacetime Poincaré symmetry and worldsheet conformal symmetry written in terms of oscillator coordinates on phase space, whose Poisson bracket is simply

$$[\alpha_m^\mu, \alpha_n^\nu]_{\text{P.B.}} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{\text{P.B.}} = im\eta^{\mu\nu}\delta_{m+n,0} \ , \qquad [p^\mu, x^\nu]_{\text{P.B.}} = \eta^{\mu\nu} \ .$$

(*a*) The spacetime Poincaré symmetry charges for the closed string are written in terms of oscillator coordinates according to

$$\begin{aligned} P^\mu &= p^\mu \ , \\ M^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty} \frac{\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu}{n} - i\sum_{n=1}^{\infty} \frac{\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu}{n} \ . \end{aligned}$$

Compute the Lie algebra of these charges.

(*b*) Derive the expression for the conserved charges L_m and $\tilde{L}_m$ used to implement the stress tensor constraints on the string phase space. Verify that their Lie algebra is the Witt algebra,

$$[L_m, L_n]_{\text{P.B.}} = i(m-n)L_{m+n} \ . \tag{1}$$

Show that the individual oscillator coordinates transform under the action of these charges according to

$$\begin{aligned} [L_m, \alpha_n^\mu]_{\text{P.B.}} &= -in\alpha_{m+n}^\mu \ . \\ [L_m, \tilde{\alpha}_n^\mu]_{\text{P.B.}} &= [\tilde{L}_m, \alpha_n^\mu]_{\text{P.B.}} = 0 \ . \end{aligned} \tag{2}$$

Deduce the action of the L_m and $\tilde{L}_m$ on the spacetime coordinate fields $X^\mu(\tau, \sigma)$. Show that these agrees with the action of the vector fields generating worldsheet conformal transformations, $V_m^\pm = \frac{1}{2}e^{-2im\sigma^\pm}\partial_\pm$.

2. Charge algebras for the quantized theory

Upon quantization, the Poisson brackets for the oscillator coordinates are promoted to commutation relations for corresponding creation and annihilation operators (and zero-mode operators) acting on the string Fock space,

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \ , \qquad [p^\mu, x^\nu] = -i\eta^{\mu\nu} \ , \tag{3}$$

where $\alpha_{-m}^\mu = (\alpha_m^\mu)^\dagger$ and similarly for the $\tilde{\alpha}$'s. In this exercise, you will study the Fock space operators that correspond to the various conserved charges encountered in our analysis of the classical string.

(*a*) Write the generators of spacetime Poincaré symmetry in terms of oscillator creation and annihilation operators. By direct computation or otherwise, show that the commutation relations for these charges are those of the Poincaré algebra computed in Q1a.

(*b*) In the quantum theory, the worldsheet conformal generators are given be

$$
\begin{aligned}
L_m &= \tfrac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{m-k} \cdot \alpha_k \ , \\
L_0 &= \tfrac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k \ .
\end{aligned}
\tag{4}
$$

Argue that the commutation relations of these operators must take the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0} \ , \tag{5}$$

where $A(m)$ is a c-number (*i.e.*, just a number) depending on m. Now determine the form of $A(m)$, either by brute force, or by arguing as follows:

(*i*) First explain why $A(-m) = -A(m)$.

(*ii*) Now using the Jacobi identity for the commutator algebra, show that for $k+m+n=0$, one has

$$(n-m)A(k) + (k-n)A(m) + (m-k)A(n) = 0 \ . \tag{6}$$

(*iii*) Based on this, show that in general the c-number term takes the form

$$A(m) = c_1 m + c_3 m^2 \ , \tag{7}$$

where c_1 and c_3 are constant c-numbers.

(*iv*) By evaluating the expectation value of $[L_m, L_{-m}]$ in the oscillator ground state $|0;0\rangle$ for $m=1$ and $m=2$, determine the values of c_1 and c_3.

3. Norms in Fock space

Compute the norms of all states at level one in the naive open-string Fock space. For which values of the normal ordering constant a will the spectrum of level-one states obeying the Virasoro constraint $(L_0 - a)|\phi\rangle = L_1|\phi\rangle = 0$ be free of ghosts?

($\star$ Bonus problem $\star$)

In class, we showed that the L_m are conserved charges on the classical string worldsheet, so

$$\frac{\partial}{\partial\tau} L_m = 0 \ .$$

On the other hand, time evolution on the string worldsheet is governed by the worldsheet Hamiltonian, which is given by $H = L_0 + \tilde{L}_0$, from which we would derive

$$\frac{\partial}{\partial\tau} L_m = [H, L_m]_{P.B.} = -imL_m \ .$$

Find the origin of this apparent discrepancy and explain why it is not a real problem.

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